An initial corrector using H2 Approach for Youla Parametrisation via LMI Optimization

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Abstract - This paper presents an approach by multiobjective optimization of the output feedback design in discrete time. The objective is to search a controller stabilizing the system with scheduled charges temporal or frequential constraints. This is achieved by using the Youla parametrization based on initial corrector H2, combined with different Lyapunov functions; via LMI (Linear Matrix Inequality) optimization a comparison of approach is done with a initial corrector LQG, another goal of this work is also reducing the conservatism.

Index Terms - Youla Parametrization, LMI Optimization, H2 control, Multiobjective control.

I. INTRODUCTION

A LMI is a constraint of affining on the design variables, the characteristics of attenuation such as the placement of poles, robust stability, execution LQG, or of RMS, gains which can be expressed like LMI. These characteristics define a multiobjective problem that can be solved numerically via convex optimization under LMI constraints[1]. The LMI optimization treats a problem with contradictory objectives, our objective is to find an optimal solution who is a compromise between all the defined objectives. The method of synthesis presented rests on the Youla parametrization [2,3]. Indeed, we use the fundamental properties of the Q-Parameter to present a methodology to obtain a representation of the inter-connected systems $G$, $J$ and $Q$ of closed loop $F_L(P,K)$ (linear Fractional Transformation or LFT Lower). We consider the H2 controller as an initial corrector for the Youla parametrization, Where the Q-Youla gives access to all the correctors $K$ who stabilize the closed loop via the parameter $Q$. Hence it exist a corrector satisfying the schedule of conditions, we can find it by convex optimization[4]. This parameterization transforms the initial problem into a convex LMI problem. This formalism is particularly adapted to the multi-criteria design because it possible to juxtapose the criteria without losing convexity [1,5,6]. In this work we can following this three steps to treat a problem of control by LMI convex optimization. The first one is the formulation of the initial problem to a optimization problem. The second is how to get a convex formulation, and the last one is the construct of the command law. Each stage of this process modifies the initial problem, and so induced a difference between the practical solution (found) and the theoretical optimal solution. Then the notion of the conservatism (Complexity / Calculability)[7] of the problem became another problem. The principle of multiobjective is to satisfy several criteria simultaneously.

II. OPTIMIZATION PROBLEM

Is defined by:

$$\min \gamma_1, \gamma_2 [\|P*K\|_{\gamma_1} < 1 \text{ et } \|P*K\|_{\gamma_2} < 1]$$

3. DEFINITION OF SOME CRITERIA

A. $H\infty$ Norm [6,7]

Matrix characterization of the $H\infty$ is represented by the inequality:

$$\exists X_1 = X_1^T > 0, \begin{bmatrix} -X_1^{-1} & A & B & 0 \\ A^T & -X_1 & 0 & C^T \\ B^T & 0 & -Y_1 & D^T \\ 0 & C & D & -Y_1 \end{bmatrix} < 0$$

B. H2 Norm [1,4,7]

Is represented by the inequality: $\exists X_2 = X_2^T$ and $Y = Y^T > 0$ such as:

$$\begin{bmatrix} -X_2^{-1} & A & 0 \\ A^T & -X_2 & C^T \\ 0 & C & -1 \end{bmatrix} < 0, \begin{bmatrix} -X_2^{-1} & B & 0 \\ B^T & -Y_1 & D^T \\ 0 & D & -Y_1 \end{bmatrix} < 0$$

$$\text{trace } (Y) < \gamma_2$$

C. Property of $\alpha$-stability [7]

A system (A,B, C, D) is $\alpha$-stable if and only if:

$$\exists X_3 = X_3^T > 0, \begin{bmatrix} \alpha & 2X_3 & A^T \\ A & X_3 & 1 \end{bmatrix} > 0$$
III. Problem of LMI Optimization [8,9]

$$\min_{\xi \in C} f(\xi)$$

$$C = \left\{ \xi \in \mathbb{R}^m \mid \forall x \in \mathbb{R}^n, x^T F(\xi) x \geq 0 \right\}$$

$$F(\xi) = F_0 + \sum_{i=1}^m \xi_i F_i$$

Some tools for LMI representation (formulation)

A. Lemma of Schur

The two inequalities are equivalent

$$\left[ A(x) > 0 \quad C(x)-B(x)^T A(x)^{-1} B(x) > 0 \right] \Rightarrow \left[ A(x) B(x) \quad B(x)^T C(x) \right] > 0$$

B. Modification by congruence

If $$A \succ 0$$ then $$\Pi^T A \Pi \succeq 0$$

C. Lemma S-procedure

Is defined by this bijective relation:

$$\begin{bmatrix} W_i \quad Z_i \end{bmatrix} \mapsto \begin{bmatrix} R_i \quad S_i \end{bmatrix} = \begin{bmatrix} W_i^{-1} & -W_i^{-1}Z_i \\ -Z_i W_i^{-1} Y_i & -Z_i W_i^{-1} Z_i \end{bmatrix}$$

IV. Multicriteria Problem

By using the following notations that refer to the figure 1, the noted transfers of a $$T_i : w_i \rightarrow z_i$$ have as a representation of state [1]:

$$T_i = \begin{bmatrix} A_{i,1} & B_{i,1} & C_{i,1} & D_{i,1} \\ C_{i,1} & A_{i,2} & B_{i,2} & D_{i,2} \end{bmatrix}$$

Then closed loop system $$T = (P*K) = \text{LFT}(P,K) = T_i(P,K)$$ check the three following properties:

$$\|T_1\|_{\infty} \leq \gamma_1 \|T_2\|_{\infty} \leq \gamma_2 \|T_3\|_{\infty} \leq \gamma_2 \text{ LFT}(P,K) \text{ is } \alpha \text{-stable}$$

if and only if there are 4 matrices $$X_1, X_2, X_3, X_4$$ such as:

$$[1,6,7]$$

$$H_\infty \begin{bmatrix} -X_1 & X_1 & A_{i,1} & X_1 & B_{i,1} & C_{i,1} & D_{i,1} \\ 0 & -X_1 & 0 & C_{i,1} & D_{i,1} \end{bmatrix} < 0$$

$$\begin{bmatrix} -X_2 & A_{i,2} & X_2 & 0 & C_{i,2} & D_{i,2} \end{bmatrix} < 0$$

V. Optimization of the Parameter of Youla

The standard representation is represented in figure 2 [2, 3]

$$\begin{bmatrix} -X_2 & A_{i,2} & X_2 & 0 & C_{i,2} & D_{i,2} \end{bmatrix} < 0$$

$$\text{trace}(\tilde{W}) < \gamma_2$$

$$\alpha\text{-stability} \begin{bmatrix} \alpha X_3 & A_{i,3} \quad X_3 \end{bmatrix} > 0$$

The inequalities of the multiobjective problem (11-13) are not linear on the unknown variables, and there does not exist today of methods to solve this kind of problem. It is thus necessary to transform it, while preserving its characteristics. We keep the Lyapunov function considered different for each criteria ($$X_i, i=1,2,3$$ for $$H_2, H_\infty, \alpha\text{-stability}$$).

Fig.2. General form of the parameterization of Youla

$$P = \begin{bmatrix} A & B_1 & D_{11} & D_{12} \\ C_1 & D_{21} & D_{22} & D_{23} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \end{bmatrix}$$

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

Defined the two dual algebraic Riccati equations [10,11]

$$A^T X_1 + X_2 A - X_2 B_2 B_2^T X_2 + C_1^T C_1 = 0$$

$$A Y_2 + Y_2 A^T - Y_2 C_2^T C_2 Y_2 + B_1 B_1^T = 0$$
The two gains $F_2$ and $L_2$ such as $(A, B_2)$ is stabilizable and $A(C_2)$ is detectable, or there the two evolution matrices $(A-B_2F_2)$ and $(A-L_2C_2)$ (18) are stable, we can computed by:

$$F_2 = -B_2^t X_2$$ \hspace{1cm} (19)

and

$$L_2 = -Y_2 C_2^t$$ \hspace{1cm} (20)

The $H_2$ corrector is defined by:

$$J_2 = \begin{bmatrix} A_2 + B_2 F_2 + L_2 C_2 & -L_2 & B_2 \\ -C_2 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$ \hspace{1cm} (21)

According to the fundamental property of the Youla parametrization $G_{22} = 0$ which expresses that all the transfer of the closed loop system are linear on $Q$ [2,3] :

$$T = P*K = P*Q = G*Q$$ \hspace{1cm} (22)

$$= P*J*Q = G_1 + G_2 (I - G_2 Q)^{-1} G_2 = G_1 + G_2 * Q * G_{21}$$

It possible to obtain a representation of state which is simultaneously in form commendable and observable in the following form [2,3] :

$$G = P*J_2$$

$$G = \begin{bmatrix} A_1 & A_2 & B_{11} & B_{21} & B_d \\ 0 & A_2 & B_{12} & B_{22} & 0 \\ C_{11} & C_{12} & D_{11} & D_{12} & D_{1u} \\ C_{21} & C_{22} & D_{21} & D_{22} & D_{2u} \\ 0 & C_{1j} & D_{j1} & D_{j2} & 0 \end{bmatrix}$$ \hspace{1cm} (23)

To obtain the Youla parameterization returns to calculate: The system of interconnection $J$ and $Q$ such as $J*Q=K$ figure 2 stabilizing for $P$, for our approach is to fix $J$ and find $Q$ optimal via LMI optimization.

VI. LINEARIZATION OF THE MATRIX INEQUALITIES

The system $G$ of the figure 2 is asymptotically stable, and $Q$ a static output-feedback. We will show that there is a LMI formulation with the control problem $H_2$, $H_\infty$ and $\alpha$-stability. This result is obtained by matrix handling. Let us consider the matrix characterizations of the $H_2$ and $H_\infty$ norms of the closed loop system $T$ (24); taking into account property 3 of the Youla parametrization, the representation of state of the system $T$ has the following form: [3]

$$T = \begin{bmatrix} A & A + B_2 Q C & B_{11} + B_2 Q D & B_{21} + B_2 Q D \\ 0 & A & B_2 & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} & D_{11} \\ C_{21} & C_{22} & D_{21} & D_{22} & D_{21} \\ 0 & C_{1j} & D_{j1} & D_{j2} & 0 \end{bmatrix}$$ \hspace{1cm} (24)

Characterizations by matrix inequalities of the $H_\infty$ and $H_2$ norms exposed to § 4 are applied to $T$ and they are not linear on the decisions variables (Xi Lyapunov functions and the gain $Q$). For example the term $X_1 A_{cl}$ (11) fact of intervening a term where intervenes in the same product $X_1$ and $Q$. It is thus necessary to modify the problem to transform it into a convex problem of optimization. So the $X_i$ is the Lyapunov function associate on the criteria $i$ (i=1,2,3 respectively $H_\infty$, $H_2$, $\alpha$-stability) it partitioned in the same proportions that the evolution matrix $A_i$:

$$X_i = [W_i, Z_i] \text{ according to } A_i = [A_1 A_2]$$ \hspace{1cm} (25)

Using the S-procedure lemma (bijective change of variable), and the propriety of congruence lemma with taking the following matrix:

$$M = \begin{bmatrix} R & 0 \\ S & I \end{bmatrix}$$ \hspace{1cm} (26)

By applying these successions of stages to the system $T$. The matrices characterizations of the three criteria described to the §4, we obtain the following LMI:

A. Problem of $H_\infty$

By using the congruence: $\Pi_1 = \text{diag}(M_1 M_1 I)$ \hspace{1cm} (27)

We obtain from (11):

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \succ 0$$

B. Problem of $H_2$

By using the congruence lemma by the two matrix: $\Pi_{21}, \Pi_{22}$ \hspace{1cm} (29)

$$\Pi_{21} = \text{diag}(M_2 M_2 1)$$ \hspace{1cm} (30)

We obtain from (12) :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \succ 0$$

C. Problem of $\alpha$-stability

By using congruence by the matrix: $\Pi_3 = \text{diag}(1 M_3)$ \hspace{1cm} (32)

we obtain from (13)
These various inequalities (28), (31), (33) are indeed linear on the variables of decisions $R_1$, $S_1$, $T_1$, $Q$ and $\gamma_1$ for the problem $H_\infty$ and $R_2$, $S_2$, $T_2$, $Q$ and $\gamma_2$ for the problem $H_2$ and $R_3$, $S_3$, $T_3$, $Q$ for the problem $\alpha$-stability. The three problems $H_2$ and $H_\infty$ and $\alpha$-stability are coupled by the static output feedback $Q$ and the different Lyapunov functions.

**Note:**

The value of $\alpha$-stable was selected to force the dominant poles of the closed loop. The sensitivity is defined by $S=(I+KG)^{-1}$ [3].

### VII. APPLICATION

We consider the system $P$ defined by : [12]

\[
P(z) = \frac{0.2879z^2 + 0.03516z - 0.2217}{z^3 - 2.158z^2 + 1.874z - 0.6908}
\]  
(35)

**A. Schedule of conditions**

1. Response time $t_{\text{r}}<1.8$ s, and time of rejection of disturbance $t_{\text{rd}}<1.8$ s.
2. Max of sensitivity function disturbance -output $|S|<15$ db.

**B. Implementation by the introduction of the weight functions**

to achieve the desired objectives. For that, one considers figure 4 in which the error $e$ is weighted by the filter $W_3(p)$, the control $u$ by $W_2(p)$, and the entry of disturbance $b$ is weighted by the filter $W_1(p)$, one can puts the whole in the following form:

\[
\begin{bmatrix}
\alpha^2 R_1 & * & * & * \\
0 & \alpha^2 T_1 & * & * \\
A R_1 & A S_1 A + B Q G + A_1 & R_1 & * \\
0 & T_2 A_2 & 0 & T_3
\end{bmatrix} > 0
\]

(33)

While taking:

\[
\begin{align*}
w_1 &= 0.82\frac{1+0.08z}{1+0.09z} \\
w_2 &= 90\frac{1+0.086z}{1+0.096z}
\end{align*}
\]

\[w_3 = 675e^{-5} \frac{1-e^{-370z}}{1+0.09z}\]

**Note.1.**

The frequential response for each functions $S$ and $KS$ is a constraint, which depends on the filter selected:

\[
\begin{align*}
&\|W_1S\|_{\infty} < \gamma \iff \forall \omega \in \mathbb{R} \left|S(j\omega)\right| < \frac{\gamma}{\|W_1(j\omega)\|} \\
&\|W_2KS\|_{\infty} < \gamma \iff \forall \omega \in \mathbb{R} \left|K(j\omega)S(j\omega)\right| < \frac{\gamma}{\|W_2(j\omega)\|}
\end{align*}
\]

(36)

**Note.2.**

To reduce the conservatism from the complexity (and calculability) point of view of the problem one satisfies to optimize $\gamma$ such as $\gamma = \gamma_1 + \lambda \gamma_2^2$ / $\lambda = 0.01 \in [0, 1]$ without losing the convexity of LMI problem. The following results are obtained:

<table>
<thead>
<tr>
<th>Recapitulative</th>
<th>With $H_2$ initial corrector</th>
<th>With LQG initial corrector</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nbr of matrix inequalities</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Nbr of decision variables</td>
<td>195</td>
<td>195</td>
</tr>
<tr>
<td>Nbr of objectives</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lyapunov $F^\infty$</td>
<td>Different</td>
<td>Different</td>
</tr>
<tr>
<td>Structure of Lyapunov $F^\infty$</td>
<td>Toeplitz</td>
<td>Toeplitz</td>
</tr>
<tr>
<td>$\alpha$-stable value</td>
<td>0.88</td>
<td>0.88</td>
</tr>
<tr>
<td>Response time</td>
<td>$t_{\text{r}}=1.5$ sec</td>
<td>$t_{\text{r}}=1.8$ sec</td>
</tr>
<tr>
<td>Rejection of disturbance time</td>
<td>$T_{\text{dist}}=1.5$ sec</td>
<td>$T_{\text{dist}}=1.5$ sec</td>
</tr>
<tr>
<td>Overschook</td>
<td>$D=1.10$ db</td>
<td>$D=1.20$ db</td>
</tr>
<tr>
<td>Time computing LMI</td>
<td>47.5101 sec</td>
<td>8.8001 sec</td>
</tr>
<tr>
<td>Max of the sensitivity $S$</td>
<td>$</td>
<td>S</td>
</tr>
<tr>
<td>$Q_{\text{opt}}$</td>
<td>$Q_{\text{opt}}=4.0082$</td>
<td>$Q_{\text{opt}}=0.9551$</td>
</tr>
<tr>
<td>$\gamma_{\text{opt}}$</td>
<td>$\gamma_{\text{opt}}=2.9162$</td>
<td>$\gamma_{\text{opt}}=1.6327$</td>
</tr>
</tbody>
</table>

![Fig.4. Block diagram of the augmented system](image1)

![Fig.5. Response of the system with LQG corrector initial](image2)
From the properties of the Youla parametrization we can to present a methodology who it possible to obtain a representation of the inter-connected systems of the closed loop. This parametrization can to ensure also the convexity of the problem (optimal solution), under an initial H2 corrector which gave us a good results. The solution given by the LMI optimization for the defined objectives is a compromise between all objectives, and this design in term of optimization is similar to the approach of the optimality defined by Pareto. Moreover the unacceptable computing time and the occupied memory capacity increase the problem of conservatism. The quality of the results (responses) depends on the initial corrector selected. In this work the optimal response of the closed loop system was enhancing by ours initial corrector.

REFERENCES

[1]. A. Molina-Cristobal, I.A. Grifin, P.J. Fleming and D.H. Owens, Linear matrix inequalities and evolutionary optimization in multiobjective control, IJSS, V.37, N°8.20, June 2006
[6]. Fuzhong Wang, Qingling Zhang, LMI-Based reliable H∞ filtering with sensor failure, IJCIC, V2, N°4, August 2006.
[7]. B. Clement, G. Duc; Multiobjc tive Control via youla parametrisation and LMI optimisation : Application to flexile arm, IFAC Symposium on Robust Control and Design, Prague, juin (2000)
[8]. Pascal Gahinet , Arkadi Nemirovski, Alan J. Laub, Mahmoud Chilali; LMI Control Toolbox Ver. 3. 2001