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Contents

Introduction 2

1 Markov semigroups in $\mathbb{R}^N$ 9
   1.1 The transition kernel, some properties .................. 13
   1.2 The weak generator of $T(t)$ ........................... 21

2 Local regularity and integrability of transition kernels 23
   2.1 Local regularity of transition kernels .................. 23
   2.2 Integrability of transition kernels ..................... 26

3 Uniform and pointwise bounds on transition kernels 32
   3.1 Uniform bounds on transition densities .................. 32
   3.2 Pointwise bounds on transition kernels .................. 34
   3.3 Regularity properties ............................... 40

4 Time dependent Lyapunov functions and kernel estimates 42
   4.1 Integrability of transition kernels .................... 42
   4.2 Pointwise bounds on transition kernels .................. 45

Appendix 53
Abstract

We study global regularity properties of transitions kernels associated with second order differential operators in $\mathbb{R}^N$ with unbounded drift and potential terms. Under suitable conditions, we prove Sobolev regularity of transition kernels and pointwise upper bounds. We use time dependent Lyapunov function techniques allowing us to gain a better time behaviour of such kernels. As an application, we obtain sufficient conditions implying the differentiability of the associated semigroup on the space of bounded and continuous functions on $\mathbb{R}^N$.

Keywords: Semigroups, transition kernels, parabolic regularity, Lyapunov functions.

Résumé

Nous étudions les propriétés de la régularité du nayou de transition associé à un opérateur différentiel du second ordre dans $\mathbb{R}^N$ à coefficients drift et potential non bornés. Sous des conditions convenables, nous montrons la régularité de Sobolev du nayou de transition et les estimations supérieures ponctuelles. En utilisons la technique des fonctions de Lyapunov dépendent du temps pour avoir des estimations optimales du nayou. Comme application nous obtenons des conditions suffisantes pour avoir la différentiabilité du semi-groupe associé dans l’espace des fonctions continue borné dans $\mathbb{R}^N$.

Mots clé: Semi-groupes, Nayou de transition, Régularité parabolique, Fonctions de Lyapunov.
Acknowledgements
Introduction

Elliptic operators with bounded coefficients have been widely studied in the literature both in $\mathbb{R}^N$ and in open subsets of $\mathbb{R}^N$, starting from the 1950’s, and nowadays they are well understood. In the last years, owing to their connections with probability and stochastic analysis, there has been an increasing interest towards linear elliptic and parabolic operators with unbounded coefficients. In literature, one can find a careful theory concerning solutions of Cauchy problems associated with the above mentioned operators in several function spaces. Many aspects such as existence, uniqueness, regularity, integral representation are object of study for numerous authors. We will deal with elliptic operators of form

$$Au(x) = \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_ju(x)) + \sum_{i=1}^{N} F_i(x)D_iu(x) - V(x)u(x).$$

with $(a_{ij})$ symmetric matrix satisfying the ellipticity condition, $a_{ij}, F_i, V$ real-valued functions, $V$ positive potential. Under Hölderianity assumptions on the coefficients, an existence result for bounded classical solutions of the Cauchy problem

$$\begin{cases} u_t(x,t) = Au(x,t), & x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}^N \end{cases}$$

with initial datum $f \in C_0(\mathbb{R}^N)$ holds (see [2]). The solution is constructed through an approximation procedure as the limit of solutions of Cauchy-Dirichlet problems in suitable bounded domains and is given by a certain semigroup $T(t)$ applied to the initial datum $f$. Moreover it can be represented by the formula

$$u(x,t) = \int_{\mathbb{R}^N} p(x,y,t)f(y)dy, \ t > 0, \ x \in \mathbb{R}^N,$$
where $p$ is a positive function called integral kernel. In this work, our attention is mainly devoted to the study of the integral kernel $p$, in particular we prove upper bounds on these kernels. Here $p$ is a positive function and for almost every $y \in \mathbb{R}^N$, it belong to $C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(\mathbb{R}^N \times (0,\infty))$ as a function of $(x,t)$ and solves the equation $\partial_t p = A p$, $t > 0$. We refer to [2, chapter 1] and [25] (in the case $V = 0$) for a review of these results as well as for conditions ensuring uniqueness for the Cauchy problem. Now, we fix $x \in \mathbb{R}^N$ and consider $p$ as a function of $(y,t)$, then $p$ satisfies

$$\partial_t p = A^* p, \ t > 0$$

where $A^*$ denotes the adjoint operator of $A$, which acts on the variable $y$. In the following sense, (see [23]) let $0 \leq t_1 \leq t_2$ and $\varphi \in C^{2,1}(Q(t_1,t_2))$ be such that $\varphi(.,t)$ has compact support for every $t \in [t_1,t_2]$. Then

$$\int_{Q(t_1,t_2)} (\partial_t \varphi(y,t) + A \varphi(y,t)) p(x,y,t) dy dt = \int_{\mathbb{R}^N} p(x,y,t_2) \varphi(y,t_2) - p(x,y,t_1) \varphi(y,t_1) dy.$$

The semigroup associated with the Schrödinger operator characterized by a vanishing drift term ($F = 0$) can be built under weaker assumptions on the potential by means of the quadratic form method. It is sufficient the requirement $V \in L^1_{loc}(\mathbb{R}^N)$ to obtain a strongly continuous analytic semigroup on $L^2(\mathbb{R}^N)$ that can be extrapolated to $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ and that admits an integral representation. For example, If $A$ is given by $\Delta - V$, the kernel $p$ is pointwise dominated by the heat kernel of the Laplacian in $\mathbb{R}^N$, that is

$$p(x,y,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \exp(-\frac{|x-y|^2}{4t}), \ (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

Kolmogorov operators, that is elliptic operators with unbounded drift term and vanishing potential, have also been studied. Some results concerning pointwise upper bounds for their kernels can be found for example in [23] where the authors use Lyapunov functions techniques to prove upper bounds for $p$ of the form $p(x,y,t) \leq c(t)\omega(y)$. Under the assumption that $F \in W^1_{\infty,loc}(\mathbb{R}^N,\mathbb{R}^N), |F|^k p, |\text{div} F|^\frac{k}{2} p \in L^1(\mathbb{R}^N \times (a,T))$ with $k > 2(N+2)$, the uniform upper bounds on $|D_yp|$ are obtained. Analogously, in the case where $F$ and its derivatives up to the second order satisfy growth conditions
of exponential type, upper bounds are also obtained for $|D_{yy}p|$ and $|\partial_t p|$. In recent papers (see [6], [7]), Bogachev, Krylov, Röckner and Shaposhnikov prove regularity and pointwise estimates for parabolic problems having measures as initial data, they also deduce uniform boundedness of solutions but we cannot compare their estimates with our results since the fundamental solution $p$ is singular for $t = 0$.

The aim of this work is to study global regularity properties of the kernel $p$ as a function of $(y,t) \in \mathbb{R}^N \times (a,T)$ for $0 < a < T$, starting from the existence of the Markov semigroup associated in $C_b(\mathbb{R}^N)$ with the operator $A$. In the case when the coefficients of the operator $A$ are bounded, the natural way to construct analytically such a semigroup consists in defining, for any $f \in C_b(\mathbb{R}^N)$ and any $t > 0$, $T(t)f$ as the value at $t$ of the classical solution of the Cauchy problem. In the first chapter we give some notation and we introduce the generalization procedure to the case when the coefficients are unbounded, we are led to prove existence results for the bounded classical solution of the Cauchy problem. Here, by bounded classical solution, we mean a function $u$ which is bounded and continuous in $\mathbb{R}^N \times [0, +\infty)$ and admits first-order time derivative and first and second-order spatial derivatives, which are continuous in $\mathbb{R}^N \times (0, +\infty)$. The arguments used to prove the existence of a classical solution are very simple. They are based both on an approximation argument with Cauchy-Dirichlet problems in bounded and smooth domains, and classical Schauder estimates. In general, this classical solution is not unique to the Cauchy problem, this is a typical feature of elliptic operators with unbounded coefficients. Nevertheless, $u_f$ enjoys a nice property: when $f \geq 0$, $u_f$ is the minimal positive solution to the Cauchy problem. This minimality property allows us to define the semigroup $T(t)$ by setting $T(t)f = u_f(\cdot, t)$. In general, such a semigroup is neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$. In fact, $T(t)f$ converge to $f$ as $t$ goes to 0, uniformly on compact subsets, but in general, not uniformly in $\mathbb{R}^N$, and this happens even if $f$ is uniformly continuous.

In chapter 2, we prove that $p(x, \cdot, \cdot)$ belong to $W^{1,0}_k(\mathbb{R}^N \times (0, T))$ provided that

$$\int_{a_0}^T \int_{\mathbb{R}^N} (V^k(y) + |F(y)|^k)p(x, y, t)dydt < \infty, \quad \text{for all} \quad k > 1$$

for fixed $x \in \mathbb{R}^N$ and $0 < a_0 < a$. This generalizes [22, Corollary 3.1, Lemma 3.1] and in some sense [4, theorem 4.1]. Assuming that certain Lyapunov functions (exponentials or powers) are integrable with respect to $p(x, y, t)dy$
for \((x,t) \in \mathbb{R}^N \times (0,T)\), pointwise upper bounds for \(p\) are obtained in chapter 3 and if in addition \(V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N), F \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)\) are such that \(DV, DF\) are dominated by some exponential functions, then \(p \in W^{2,1}_k(\mathbb{R}^N \times (0,T))\) for all \(k > 1\). As a consequence, we obtain also upper bound for \(|D_y p|\). In the case where \(F\) and \(V\) and their corresponding derivatives up to the second order satisfy growth conditions of exponential type, upper bounds are also obtained for \(|D_{yy} p|\) and \(|\partial_t p|\). As a consequence, we deduce that the semigroup \(T(\cdot)\) is differentiable on \(C^b(\mathbb{R}^N)\) for \(t > 0\). In the case where \(V = 0\), regularity and pointwise estimates for \(p\) can be found in [23], [29] and for the solution of the Cauchy problem with a \(L^1\) function as the initial datum we refer to [6], [7]. In chapter 4 we prove pointwise bounds for the transition density \(p\) as a function of \((y,t) \in \mathbb{R}^N \times (a,T)\) for \(0 < a < T\), using time dependent Lyapunov functions for the parabolic operator \(L = \partial_t + A\), we prove better and more general time estimates for the transition kernel \(p\), compared to those obtained in chapter 3 and [29]. As an application, we show that the kernel \(p\) associated to the operator \(A = \Delta - |x|^{r-1}x.D - |x|^s\) with \(r > 1\) and \(s > \max\{4,2r\}\) satisfies

\[
0 < p(x,y,t) < \frac{C}{t^{\frac{N}{2}}} \exp(-\gamma |tx|^{1+\frac{s}{2}}) \exp(-\gamma |ty|^{1+\frac{s}{2}})
\]

provided that \(\gamma < \frac{1}{2+s}\) and \(k > N + 2\). For Kolmogorov operators with drift term of gradient type, kernel estimates can be found in [8, section 4.7]. Such operators are similar to Schrödinger operators. Kernel estimates for Schrödinger operators with potentials of polynomial type are proved in [8, section 4.5].
Notation

Sets

\( \mathbb{N} \) set of all positive natural numbers.
\( \mathbb{R} \) set of all real numbers.
\( \mathbb{R}^N \) set of all real N-tuples.
\( B(x, R) \) denotes the open ball of \( \mathbb{R}^N \) of radius \( R \) and center \( x \).
\( \overline{B}(x, R) \) denotes the closure of \( B(x, R) \).
For \( 0 \leq a < b \), we use \( Q(a, b) \) for \( \mathbb{R}^N \times (a, b) \) and \( Q_T \) for \( Q(0, T) \) (here the intervals can be either open or closed).
\( C = C(a_1, \ldots, a_n) \) write to point out that the constant \( C \) depends on the quantities \( a_1, \ldots, a_n \). To simplify the notation, we understand the dependence on the dimension \( N \) and on quantities determined by the matrix \((a_{ij})\) as the ellipticity constant or the modulus of continuity of the coefficients.

Functions

\( \delta_x \) the delta function, i.e., \( \delta_x(x) = 1, \delta_x(y) = 0 \) if \( y \neq x \).
\( \chi_A \) the characteristic function of the set \( A \), i.e., the function defined by \( \chi_A(x) = 1 \) for any \( x \in \mathbb{R}^N \) and \( \chi_A(x) = 0 \) for any \( x \notin A \).
\( 1 \) the characteristic function of \( \mathbb{R}^N \).

If \( u : \mathbb{R}^N \times J \to \mathbb{R} \), where \( J \subset [0, \infty) \) is an interval, we use:

\[
\begin{align*}
\partial_t u &= \frac{\partial u}{\partial t}, \quad D_i u = \frac{\partial u}{\partial x_i}, \quad D_{ij} u = D_i D_j u \\
D u &= (D_1 u, \ldots, D_N u), \quad D^2 u = (D_{ij} u)
\end{align*}
\]

and

\[
|Du|^2 = \sum_{j=1}^N |D_j u|^2, \quad |D^2 u|^2 = \sum_{i,j=1}^N |D_{ij} u|^2.
\]

Function spaces

\( C^j_b(\mathbb{R}^N) \) is the space of \( j \) times differentiable functions in \( \mathbb{R}^N \), with bounded
derivatives up to the order $j$. $C^\infty_c(\mathbb{R}^N)$ is the space of test functions. $C^\alpha(\mathbb{R}^N)$ denotes the space of all bounded and $\alpha$-Hölder continuous functions on $\mathbb{R}^N$.

For $1 \leq k \leq \infty$, $j \in \mathbb{N}$, $W^j_k(\mathbb{R}^N)$ denotes the classical Sobolev space of all $L^k$-functions having weak derivatives in $L^k(\mathbb{R}^N)$ up to the order $j$. Its usual norm is denoted by $\| \cdot \|_{j,k}$ and by $\| \cdot \|_k$ when $j = 0$.

Let us now define some spaces of functions of two variables following basically the notation of [19]).

$C_0(Q(a,b))$ is the Banach space of continuous functions $u$ defined in $Q(a,b)$ such that $\lim_{|x| \to \infty} u(x,t) = 0$ uniformly with respect to $t \in [a,b]$.

$C^{2,1}(Q(a,b))$ is the space of all bounded functions $u$ such that $\partial_t u$, $Du$ and $D_{ij}u$ are bounded and continuous in $Q(a,b)$.

For $0 < \alpha \leq 1$ we denote by $C^{2+\alpha,1+\alpha/2}(Q(a,b))$ the space of all bounded function $u$ such that $\partial_t u$, $Du$ and $D_{ij}u$ are bounded and $\alpha$-Hölder continuous in $Q(a,b)$ with respect to the parabolic distance $d((x,t),(y,s)) := |x - y| + |t - s|^{1/2}$. Local Hölder spaces are defined, as usual, requiring that the Hölder condition holds in every compact subset.

We shall also use parabolic Sobolev spaces. We denote by $W^{r,s}_k(Q(a,b))$ the space of functions $u \in L^k(Q(a,b))$ having weak space derivatives $D^\alpha_x u \in L^k(Q(a,b))$ for $|\alpha| \leq r$ and weak time derivatives $\partial_t^\beta u \in L^k(Q(a,b))$ for $\beta \leq s$, equipped with the norm

$$
\|u\|_{W^{r,s}_k(Q(a,b))} := \|u\|_{L^k(Q(a,b))} + \sum_{|\alpha| \leq r} \|D^\alpha_x u\|_{L^k(Q(a,b))} + \sum_{|\beta| \leq s} \|\partial_t^\beta u\|_{L^k(Q(a,b))}.
$$

$\mathcal{H}^{k,1}(Q_T)$ denotes the space of all functions $u \in W^{1,0}_k(Q_T)$ with $\partial_t u \in (W^{1,0}_k(Q_T))'$, the dual space of $W^{1,0}_k(Q_T)$, endowed with the norm

$$
\|u\|_{\mathcal{H}^{k,1}(Q_T)} := \|\partial_t u\|_{(W^{1,0}_k(Q_T))'} + \|u\|_{W^{1,0}_k(Q_T)}
$$

where $\frac{1}{k} + \frac{1}{k'} = 1$. 7
Finally, for $k > 2$, $\mathcal{V}^k(Q_T)$ is the space of all functions $u \in W^{1,0}_k(Q_T)$ such that there exists $C > 0$ for which

$$\left| \int_{Q_T} u \partial_t \phi \, dx \, dt \right| \leq C \left( \|\phi\|_{L^{k-2}_{k-2}(Q_T)} + \|D\phi\|_{L^{k-1}_{k-1}(Q_T)} \right)$$

for every $\phi \in C^{0,1}_c(Q(a, b))$. Notice that $k/(k-1) = k'$, $k/(k-2) = (k/2)'$.

$\mathcal{V}^k(Q_T)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathcal{V}^k(Q_T)} = \|u\|_{W^{1,0}_k(Q_T)} + \|\partial_t u\|_{k/2,k;Q_T},$$

where $\|\partial_t u\|_{k/2,k;Q_T}$ is the best constant $C$ such that the above estimate holds.

The space $H^{k,1}_t(Q_T)$ was introduced and studied by Krylov [14]. All properties of the spaces $H^{k,1}_t(Q_T)$ and $\mathcal{V}^k(Q_T)$ needed here, can be found in Appendix. In the whole thesis the transition density $p$ will be considered as a function of $(y,t)$ for arbitrary but fixed $x \in \mathbb{R}^N$. The writing $\|p\|$ therefore stands for any norm of $p$ as function of $(y,t)$, for a fixed $x$. 
Chapter 1

Markov semigroups in $\mathbb{R}^N$

In this chapter we collect some preliminary results needed to develop the next theory. In particular we introduce elliptic operators with unbounded coefficients and we study the Markov semigroups associated with them. We consider the operator

$$Au(x) = \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j)u(x) + \sum_{i=1}^{N} F_i(x)D_i u(x) - V(x)u(x).$$

under the hypotheses the following conditions on the coefficients which will be kept without further mentioning.

(H) $a_{ij} = a_{ji}$, $F_i : \mathbb{R}^N \to \mathbb{R}$, with $a_{ij} \in C^{1+\alpha}(\mathbb{R}^N)$, $0 \leq V, F_i \in C^\alpha_{\text{loc}}(\mathbb{R}^N)$ for some $0 < \alpha < 1$ and

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i \xi_j \leq \Lambda |\xi|^2$$

for every $x, \xi \in \mathbb{R}^N$ and suitable $0 < \lambda \leq \Lambda$.

Notice that the drift $F = (F_1, \ldots, F_N)$ and the potential $V$ are not assumed to be bounded in $\mathbb{R}^N$.

Besides, we introduce the realization $\mathcal{A}$ of $A$ in $C_b(\mathbb{R}^N)$, with domain $D_{\text{max}}(A)$, defined as follows:

$$D_{\text{max}}(A) = \{u \in C_b(\mathbb{R}^N) \cap \bigcap_{1 \leq p < +\infty} W^{2,p}_{\text{loc}}(\mathbb{R}^N) : Au \in C_b(\mathbb{R}^N)\}, \mathcal{A}u = Au.$$
In the first section we prove the existence of a classical solution \( u \in C(\mathbb{R}^N \times [0, +\infty)) \cap C^{1,2}(\mathbb{R}^N \times (0, +\infty)) \) of the Cauchy problem
\[
\begin{cases}
  u_t(x,t) = Au(x,t), & x \in \mathbb{R}^N, \ t > 0 \\
  u(x,0) = f(x), & x \in \mathbb{R}^N
\end{cases}
\tag{1.0.1}
\]
which is bounded in \( \mathbb{R}^N \times [0,T] \) for any \( T > 0 \) and satisfies
\[
\partial_t u, D^2 u \in C^{2+\alpha,1+\alpha}(\mathbb{R}^N \times (0, +\infty)).
\]

Since the coefficients of the operator are not bounded, the classical theory does not give a solution of the problem. The solution is constructed through an approximation procedure as limit of solutions of Cauchy-Dirichlet problem
\[
\begin{cases}
  \partial_t u_n(x,t) = Au_n(x,t), & x \in B(0,n), \ t > 0 \\
  u_n(x,t) = 0, & x \in \partial B(0,n), \ t > 0 \\
  u_n(x,0) = f(x), & x \in B(0,n)
\end{cases}
\tag{1.0.2}
\]

in the ball \( B(0,n) \) for a given \( f \in C_c(\mathbb{R}^N) \) and \( n \in \mathbb{N} \) with \( \text{supp} f \subseteq B(0, n_0) \) and \( n \geq n_0 \). By classical results for parabolic Cauchy problems (1.0.2) in bounded domains, we know that the problem (1.0.2) admits a unique solution \( u_n \in C(B(0,n) \times [0, \infty)) \cap C^{2+\alpha,1+\alpha}(B(0,n) \times (0, \infty)) \). Using Schauder interior estimates (see Appendix, Theorem 4.2.7) and compactness argument, we prove that we can define a function \( u : \mathbb{R}^N \times (0, +\infty) \rightarrow \mathbb{R} \) by setting
\[
 u(x,t) = \lim_{n \rightarrow +\infty} u_n(x,t),
\]
for any \( t \in [0, +\infty) \) and any \( x \in \mathbb{R}^N \). Such a function belong to \( C(\mathbb{R}^N \times [0, +\infty)) \cap C^{1+\frac{\alpha}{2},2+\alpha}(\mathbb{R}^N \times (0, +\infty)) \), is a solution of the problem (1.0.2) and satisfies the estimates
\[
|u(x,t)| \leq \|f\|_\infty, \ t > 0, x \in \mathbb{R}^N.
\]

**Theorem 1.0.1** [2, Theorem, 2.2.1] For any \( f \in C_b(\mathbb{R}^N) \), there exists a solution \( u \in C(\mathbb{R}^N \times [0, +\infty)) \) of the problem (1.0.1). The function \( u \in C^{1+\frac{\alpha}{2},2+\alpha}(\mathbb{R}^N \times (0, +\infty)) \) and
\[
|u(x,t)| \leq \|f\|_\infty, \ t > 0, x \in \mathbb{R}^N.
\tag{1.0.3}
\]
Proof. We split the proof into two steps. First, we show that there exists a solution \( u \in C^{1+\frac{2}{N},2+\alpha}(\mathbb{R}^N \times (0, +\infty)) \) to (1.0.1), and it satisfies (1.0.3). Then, in step 2, we show that \( u \) is continuous up to \( t = 0 \), and \( u(., 0) = f \).

Step 1. For any \( n \in \mathbb{N} \), let \( u_n \in C(\overline{B}(0, n) \times [0, \infty)) \cap C^{1,2}(B(0, n) \times (0, +\infty)) \) be the solution of the Cauchy-Dirichlet problem (1.0.2) (see [2, Proposition C.3.2]), which is given by

\[
u_n(x, t) = (T_n(t)f)(x), t > 0, x \in B(0, n),\]

where \( T_n(t) \) is the semigroup in \( C(\overline{B}(0, n)) \) associated with the Cauchy-Dirichlet problem (1.0.2). From [2, Proposition C.3.2], for any \( n \in \mathbb{N} \) we have

\[
|u_n(x, t)| \leq \|f\|_{\infty}, t > 0, x \in B(0, n).
\]

(1.0.4)

Now, fix \( M \in \mathbb{N} \) and set \( D(M) = (0, M) \times B(0, M) \) and \( D'(M) = \left[ \frac{1}{M}, M \right] \times B(0, M - 1) \). From the interior Schauder estimate (4.2.9) we deduce that

\[
\|u_n\|_{C^{1+\frac{2}{N},2+\alpha}(D'(M))} \leq C_M \|u_n\|_{L^\infty(D(M))} \leq C_M \|f\|_{\infty},
\]

(1.0.5)

for any \( n \geq M \), where \( C_M > 0 \) is a constant independent of \( n \in \mathbb{N} \). Fix \( \beta \in (0, \alpha) \). By (1.0.5) there exists a subsequence \( \{u^{(M)}_n\} \) of \( \{u_n\} \) converging in \( C^{1+\frac{2}{N},2+\beta}(D'(M)) \) to some function \( \{u^{(M)}_\infty\} \in C^{1+\frac{2}{N},2+\alpha}(D'(M)) \). Without loss of generality we can assume that \( \{u^{(M+1)}_n\} \) is a subsequence of \( \{u^{(M)}_n\} \). Thus, the functions \( u^{(M)}_\infty \) and \( u^{(M+1)}_\infty \) coincide in the domain \( D'(M) \) and therefore, we can define the function \( u \in C^{1+\frac{2}{N},2+\alpha}(R^N \times (0, +\infty)) \) by setting

\[
u = u^{(M)}_\infty \quad \text{in} \quad D'(M).
\]

Moreover, the diagonal subsequence defined by \( \tilde{u}_n = u^{(n)}_n, n \in \mathbb{N} \), converges to \( u \) in \( C^{1+\beta,2+\beta}([T_1,T_2] \times K) \) for any compact set \( K \subset R^N \) and any \( 0 < T_1 < T_2 \). Hence, letting \( n \) go to \( +\infty \) in the differential equation satisfies by \( \tilde{u}_n \), it follows that \( u \) satisfies the equation

\[
\partial_t u(x, t) = Au(x, t), \quad x \in \mathbb{R}^N, \quad t > 0.
\]

Besides, (1.0.3) follows.

Step 2. To complete the proof we must show that \( u \in C(R^N \times [0, +\infty)) \) and \( u(x, 0) = f(x) \). For this purpose, we take advantage of the semigroup theory. In particular, we will use the representation formula of solutions.
to Cauchy-Dirichlet problems in bounded domains through semigroups. Fix $M \in \mathbb{N}$ and let $\vartheta$ be any smooth function such that

$0 \leq \vartheta \leq 1$, $\vartheta \equiv 1$ in $B(0, M - 1)$, $\vartheta \equiv 0$ outside $B(0, M)$.

For any $n > M$, let $v_n = \vartheta \tilde{u}_n$. As it is easily seen, the function $v_n$ belong to $C(B(0, M) \times [0, +\infty))$ and is the solution of the Cauchy-Dirichlet problem

$$\begin{cases}
    \partial_t v_n(x, t) = Av_n(x, t) + \psi_n(x, t), & x \in B(0, M), \ t > 0 \\
    v_n(x, t) = 0, & x \in \partial B(0, M), \ t > 0 \\
    v_n(x, 0) = \vartheta(x) f(x), & x \in B(0, M)
\end{cases}$$

where $\psi_n$ is given by

$$\psi_n = -\sum_{i,j=1}^N q_{ij} (2D_i \tilde{u}_n D_j \vartheta + \tilde{u}_n D_{ij} \vartheta) - \tilde{u}_n \sum_{i=1}^N b_i D_i \vartheta.$$

For any $t > 0$ and any $x \in B(0, M)$ we have

$$|\psi_n(x, t)| \leq K_M \|f\|_{\infty} + \sum_{i=1}^N \|D_i \tilde{u}_n(., t)\|_{L^\infty(B(0,M))},$$

where $K_M > 0$ is such that

$$\sum_{i,j=1}^N \|q_{ij} D_{ij} \vartheta\|_{L^\infty(B(0,M))} + \sum_{i=1}^N \|b_i D_i \vartheta\|_{L^\infty(B(0,M))} \leq K_M,$$

$$2 \sum_{j=1}^N \|q_{ij} D_j \vartheta\|_{L^\infty(B(0,M))} \leq K_M, \ i = 1, \ldots, N.$$

We consider again the interior estimate of Theorem 4.2.7. By (4.2.10), the function $\tilde{u}_n$ satisfies the estimate

$$|\sqrt{t} D \tilde{u}_n(x, t)| \leq C \|\tilde{u}_n\|_{L^\infty(D(M+1))} \leq C \|f\|_{\infty},$$

for any $x \in B(0, M)$, any $0 < t < 1 = dist(B(0, M), \partial B(0, M+1))$ and some positive constant $C$, independent of $n$. This yields

$$\|D_i \tilde{u}_n(., t)\|_{L^\infty(B(0,M))} \leq t^{\frac{N}{2}} C \|f\|_{\infty}, 0 < t \leq 1,$$
for any $i = 1, \ldots, N$. Then, by (1.0.6) it follows that
\[ |\psi_n(x, t)| \leq K'_M (1 + t^{-\frac{1}{2}}) \|f\|_{\infty}, t \in (0, 1], x \in B(0, M), \] (1.0.8)
for any $n > N$, where $K'_M > 0$ is a constant independent of $n$. Therefore, $\psi_n \in L^1(0, T, C(B(0, M)))$ and we can represent $v_n$ by means of variation of-constants formula
\[ v_n(t) = T_M(t)(\vartheta f) + \int_0^t T_M(t-s)\psi_n(s)ds, t > 0, \]
where, as usual, $T_M(t)$ is the semigroup associated with the operator $A$ with homogeneous Dirichlet conditions on $\partial B(0, M)$. Since $v_n \equiv \tilde{u}_n$ and $\vartheta \equiv 1$ in $B(0, M - 1)$, by (1.0.4) and (1.0.8) it follows
\[ |\tilde{u}_n(x, t) - f(x)| \leq \|T_M(t)(\vartheta f) - \vartheta f\|_{\infty} + K'_M \|f\|_{\infty} \int_0^t (1 + s^{-\frac{1}{2}})ds, \]
for any $t > 0$ and any $x \in B(0, M - 1)$. Letting $n$ go to $+\infty$ we get
\[ |u(x, t) - f(x)| \leq \|T_M(t)(\vartheta f) - \vartheta f\|_{\infty} + K'_M \|f\|_{\infty} \int_0^t (1 + s^{-\frac{1}{2}})ds, \]
which shows that $u$ is continuous at $t = 0$ and $x \in B(0, M - 1)$. Since $M \in N$ is arbitrary, we have $u \in C(\mathbb{R}^N \times [0, +\infty))$ and $u(., 0) \equiv f$. \hfill \Box

1.1 The transition kernel, some properties

We now recall the definition of transition function. Here $B(E)$ denotes the $\sigma-$algebra of Borel sets of a topological space $E$.

**Definition 1.1.1** A family of Borel measures $G(x, .): t \geq 0, x \in E$ is a transition function if the function $G(x, ., B): E \to \mathbb{R}$ is Borel measurable for any $t \geq 0$ and any $B \in B(E)$, and

(i) $G(x, E, t) \leq 1$ for any $t \geq 0, x \in E$;

(ii) $G(x, E \setminus \{x\}, 0) = 0$ for any $x \in E$;

(iii) $G(x, B, t + s) = \int_E G(y, B, s)G(x, dy; t)$ for any $s, t \geq 0, x \in E, B \in B(E)$.  

13
A transition function is normal if \( \lim_{t \to 0^+} G(x, E, t) = 1 \) for any \( x \in E \); it is stochastically continuous if for any open set \( U \subset E \) it holds that
\[
\lim_{t \to 0^+} G(x, U, t) = 1,
\]
whenever \( x \in U \).

**Theorem 1.1.2** [2, Theorem, 2.2.5] There exists a semigroup of linear operators \( T(t) \) defined in \( C_b(R^N) \) such that, for any \( f \in C_b(R^N) \), the solution of the problem (1.0.1), given by the Theorem (1.0.1), is represented by
\[
u(x, t) = (T(t)f)(x), t \geq 0, x \in R^N.
\]

(1.1.1)

For any \( t > 0 \), \( T(t) \) satisfies the estimate
\[
\|T(t)f\|_\infty \leq \|f\|_\infty, f \in C_b(R^N).
\]

(1.1.2)

Moreover, there exist a family of Borel measures \( G(x, dy; t) \) in \( R^N \) such that
\[
(T(t)f)(x) = \int_{R^N} f(y)G(x, dy; t), t \geq 0, x \in R^N,
\]

(1.1.3)

and a function \( p : R^N \times R^N \times (0, +\infty) \to R \) such that
\[
G(x, dy; t) = p(x, y, t)dy, t > 0, x, y \in R^N.
\]

(1.1.4)

The function \( p \) is strictly positive and the functions \( p(., ., t) \) and \( p(x, ., t) \) are measurable for any \( t > 0 \) and any \( x \in R^N \). Further for almost any fixed \( y \in R^N \), the function \( p(., y, .) \) belong to the space \( C^{1+\frac{\alpha}{2}, 2+\alpha}_{loc}(R^N \times (0, +\infty)) \), and it is a solution of the equation \( \partial_t u - Au = 0 \).

**Proof.** Step 1: definition and properties of \( p \). For any \( n \in N \) let \( p_n \in C(B(0, n) \times B(0, n) \times (0, \infty)) \) be the fundamental solution of the equation \( \partial_t u - Au = 0 \) in \( B(0, n) \), given by [2, Proposition C.3.2]. We extend the function \( p_n \) to \( R^N \times R^N \times (0, +\infty) \) with value zero for \( x, y \notin B(0, n) \) and still denote by \( p_n \) the so obtained function. A straightforward computation shows that for any fixed \( t \in (0, +\infty) \) and any \( x, y \in R^N \), the sequence \( p_n(x, y, t) \) is increasing. Indeed, for any positive \( f \in C(B(0, n)) \), the function
\[
w(x, t) = \int_{p(n)} f(y)(p_{n+1}(x, y, t) - p_n(x, y, t))dy, t \geq 0, x \in R^N,
\]

14
is positive. Recalling that, for any $t > 0$ and any $x \in B(0, n)$, the function $p_{n+1}(x, \ldots, t) - p_n(x, \ldots, t)$ is continuous in $B(0, n)$, we easily deduce that $p_{n+1}(x, y, t) \geq p_n(x, y, t)$ for any $t > 0$ and any $x, y \in B(0, n)$, implying that the sequence $\{p_n(x, y, t)\}$ is increasing. Hence, we can define the function $p : \mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty) \rightarrow \mathbb{R}$ by setting

$$p(x, y, t) = \lim_{n \to +\infty} p_n(x, y, t), t > 0.$$  

The function $p$ is finite almost everywhere. Indeed,

$$\int_{B(0,n)} p_n(x, y, t) dy \leq 1, t > 0, x \in B(0, n),$$

and then, by monotone convergence,

$$\int_{\mathbb{R}^N} p(x, y, t) dy \leq 1, t > 0, x \in \mathbb{R}^N, \quad (1.1.5)$$

so that $p(x, y, t)$ is finite for any $t > 0, x \in \mathbb{R}^N$ and almost $y \in \mathbb{R}^N$. Since $p_n(x, \ldots, t) > 0$ almost everywhere in $B(0, n)$ for any $t > 0$ and any $x \in B(0, n)$, then $p$ is strictly positive. Of course, $p(\ldots, t), p(x, \ldots, t)$ and $p(\ldots, y, t)$ are measurable functions for any $t > 0$ and any $x, y \in \mathbb{R}^N$ since they are the pointwise limit of measurable functions.

We now prove the regularity properties of $p$. Fix $R, T > 0, x_0 \in B(0, R)$ and let $y_0 \in \mathbb{R}^N$ be such that $p(x_0, y_0, T) < +\infty$; actually we have seen that this holds for almost any $y_0 \in \mathbb{R}^N$. If $h, n \in N$ satisfy $R + 1 < h < n$, the functions $p_h(\ldots, y_0, \ldots)$ and $p_n(\ldots, y_0, \ldots)$ are solutions of the equation $\partial_t u - Au = 0$ in $B(0, R+1) \times (0, +\infty)$ (see Theorem 4.2.7), and hence $p_h(\ldots, y_0, \ldots) - p_n(\ldots, y_0, \ldots)$ is as well. Moreover, $p_n(\ldots, y_0, \ldots) - p_h(\ldots, y_0, \ldots)$ is positive and, for any fixed $0 < t_0 < t_1 < T$, it satisfies the following Harnack inequality:

$$\sup\{p_n(x, y_0, t) - p_h(x, y_0, t), (x, t) \in \overline{B}(0, R) \times [t_0, t_1]\}$$

$$\leq \inf\{p_n(x, y_0, T) - p_h(x, y_0, T), x \in \overline{B}(0, R)\}$$

$$\leq C(p_n(x_0, y_0, T) - p_h(x_0, y_0, T)),$$

where $C > 0$ is a constant, independent of $h$ and $n$. Since $p_n(x_0, y_0, T) < +\infty$, then $\{p_n(\ldots, y_0, \ldots)\}$ turns out to be a Cauchy sequence in $C([t_0, t_1] \times \overline{B}(0, R))$. Since it converges pointwise to the function $p(x, y_0, t)$, we conclude that $p(\ldots, y_0, \ldots) \in C([t_0, t_1] \times \overline{B}(0, R))$. Moreover, from Theorem 4.2.7 it follows
that, for any $t'_0 > t_1$ and any $R' < R$, the sequence $p_n(., y_0, .)$ converge also in $C^{1+\frac{\alpha}{2}, 2+\beta}([t'_0, t'_1] \times B(0, R'))$ for any $\beta \in (0, \alpha)$. Hence, $p_n(., y_0, .) \in C^{1+\frac{\alpha}{2}, 2+\beta}([t'_0, t'_1] \times B(0, R'))$. Since $T, R, R', t_0, t_1, t'_0, t'_1 > 0$ are arbitrary, we get $p(., y_0, .) \in C^{1+\frac{\alpha}{2}, 2+\beta}([0, +\infty] \times \mathbb{R}^N)$. Finally, since $\partial_t p_n - Ap_n = 0$, as $n$ goes to $+\infty$, it follows that $\partial_t p - Ap = 0$.

**Step 2: definition and properties of $G(x; dy, t)$ and $T(t)$.** Now, for any $t > 0$ and any $x \in \mathbb{R}^N$ we define the measure $G(x; dy, t)$ by (1.1.4), while for $t = 0$ we set $G(x; dy, t) = \delta_x$. Then, for any $t > 0$, we define the operator $T(t)$ by (1.1.3). Let us prove that, for any $f \in C_b(\mathbb{R}^N)$, the solution $u$ of the problem (1.0.1) is given by (1.1.1). Indeed,

$$u(x, t) = \lim_{n \to +\infty} \int_{\mathbb{R}^N} f(y) p_n(x, y, t) dy, t > 0, x \in \mathbb{R}^N,$$

and we can split it as

$$u(x, t) = \lim_{n \to +\infty} \int_{\mathbb{R}^N} f^+(y) p_n(x, y, t) dy - \lim_{n \to +\infty} \int_{\mathbb{R}^N} f^-(y) p_n(x, y, t) dy.$$

By the monotone convergence theorem we immediately deduce that

$$u(x, t) = \int_{\mathbb{R}^N} f^+(y) p(x, y, t) dy - \int_{\mathbb{R}^N} f^-(y) p(x, y, t) dy = \int_{\mathbb{R}^N} f(y) p(x, y, t) dy,$$

that is (1.1.1).

To show that $T(t)$ is a semigroup it suffices to use the monotone convergence theorem, it follows that

$$p(x, y, t + s) = \int_{\mathbb{R}^N} p(x, y, s) p(z, y, t) dz, t, s > 0, x, y \in \mathbb{R}^N.$$

Moreover, (1.1.2) is an immediate consequence of (1.0.3) and (1.1.1).

\[ \square \]

**Remark 1.1.3** Analogous to the proof of the above theorem one see that for each fixed $x \in B(0, n)$ it holds

$$\partial_t p_n(x, y, t) = A^* p_n(x, y, t)$$
with respect to \((y, t) \in B(0, n) \times (0, \infty)\), where

\[
A^* = A_0 - F.D - \text{div}F - V
\]  

(1.1.6)

is the formal adjoint operator of \(A\), such that

\[
p_n^*(y, x, t) = p_n(x, y, t)
\]  

(1.1.7)

is the unique Green’s function of the problem

\[
\begin{cases}
\partial_t v_n(y, t) = A^* v_n(y, t), & y \in B(0, n), \ t > 0 \\
v_n(y, t) = 0, & y \in \partial B(0, n), \ t > 0 \\
v_n(y, 0) = f(y), & y \in B(0, n)
\end{cases}
\]  

(1.1.8)

One can find the proof of these statements in [11, Section 3]. The existence of \(p_n^*(y, x, t) = p_n(x, y, t)\) holds also under weaker assumptions such as \(a_{ij} \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N), F_i \in C^{1+\alpha}_{\text{loc}}(\mathbb{R}^N)\) and \(V \in C_{\text{loc}}^{\alpha}(\mathbb{R}^N)\) for all \(i, j = 1, \ldots, N\).

For the solution \(u_n\) of problem (1.0.2) we have

\[
\int_{B(0,n)} p_n(x, y, t) f(y) dy
\]

and

\[
\int_{B(0,n)} p_n(x, y, t) f(y) dy \longrightarrow f(x) \quad \text{as} \quad t \longrightarrow 0 \quad \text{for each} \quad x \in B(0, n),
\]

and for the solution \(v_n\) of problem (1.1.8) we have

\[
\int_{B(0,n)} p_n(x, y, t) f(x) dx
\]

and

\[
\int_{B(0,n)} p_n(x, y, t) f(x) dx \longrightarrow f(y) \quad \text{as} \quad t \longrightarrow 0 \quad \text{for each} \quad y \in B(0, n).
\]

**Remark 1.1.4** In [2, chapter 2], using the classical maximum principle, one obtains that the sequence \((p_n)\) is increasing with respect to \(n \in N\). One sets

\[
p(x, y, t) = \lim_{n \to \infty} p_n(x, y, t) \quad \text{pointwise for} \quad (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty),
\]

and defines the linear operator \(T(t)\) in \(C_b(\mathbb{R}^N)\), for any \(t > 0\), by setting

\[
T(t) f(x) = \lim_{n \to \infty} T_n(t) f(x).
\]
The above limit exists if \( f \geq 0 \) by monotonicity; for general \( f \), its existence is proved writing \( f = f^+ - f^- \) and using the linearity of the operators \( T_n(t) \). Clearly \( T(t) \) is a linear operator and, moreover, \( \|T(t)f\|_\infty \leq \|f\|_\infty \), \( T(t)f \geq 0 \) if \( f \geq 0 \), since this is true for \( T_n(t)f \). To show the semigroup law, consider first \( f \geq 0 \). Then

\[
T(t+s)f(x) = \lim_{n \to \infty} T_n(t+s)f(x) = \lim_{n \to \infty} T_n(t)T_n(s)f(x) \leq T(t)T(s)f(x).
\]

For every \( n_1 > 0 \) we have also

\[
T(t+s)f(x) = \lim_{n \to \infty} T_n(t)T_n(s)f(x) \geq \lim_{n \to \infty} T_{n_1}(t)T_{n_1}(s)f(x) = T_{n_1}(t)T(s)f(x)
\]

and, letting \( n_1 \to \infty \), we obtain that \( T(t+s)f(x) \geq T(t)T(s)f(x) \), whence the semigroup law for positive \( f \). The general case immediately follows.

In general, \((T(t))_{t \geq 0}\) is not strongly continuous semigroup in \( C_b(\mathbb{R}^N) \) nevertheless, \( T(t)f \) tends to \( f \) as \( t \) tends to 0, uniformly on compact sets. If \( f \) vanishes at infinity, then actually, \( T(t)f \) tends to \( f \) as \( t \) tends to 0, uniformly on \( \mathbb{R}^N \). But this does not mean that the restriction of \((T(t))_{t \geq 0}\) to \( C_0(\mathbb{R}^N) \) is a strongly continuous semigroup, because, in general, \( C_0(\mathbb{R}^N) \) is not invariant for \( T(t) \), (see [2, proposition 5.3.4]).

**Proposition 1.1.5** [2, Proposition, 2.2.7] For any \( f \in C_0(\mathbb{R}^N) \), \( T(t)f \) tends to \( f \) in \( C_b(\mathbb{R}^N) \), as \( t \) tends to \( 0^+ \).

**Proof.** We prove the statement assuming that \( f \in C_c^\infty(\mathbb{R}^N) \). The general case then will follow by density. So, let us fix \( f \in C_c^\infty(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \). Moreover, let \( k \in \mathbb{N} \) be such that \( B(0,k) \) contains both \( x \) and \( \text{supp}(f) \). Then \((A_kf)(x) = (Af)(x)\), where, as usual, \( A_k \) denotes the realization of the operator \( A \) in \( C(B(0,k)) \) with homogeneous Dirichlet conditions. Let \( u_k(t) = T_k(t)f \), where \( T_k(t) \) is the analytic semigroup generated by \( A_k \). For any \( t > 0 \) we have

\[
u_k(x,t) - f(x) = \int_0^t \frac{\partial}{\partial s} u_k(x,s)ds
\]

\[
= \int_0^t (A_kT_k(s)f)(x)ds
\]

\[
= \int_0^t (T_k(s)Af)(x)ds
\]
\[ = \int_0^t ds \int_{\mathbb{R}^N} p_k(x, y, s)Af(y)dy, \]

where we have extended \( p_k(x, \cdot, t) \) to the whole of \( \mathbb{R}^N \) by setting \( p_k(x, y, t) = 0 \) for any \( y \notin B(k) \). Letting \( k \) goes to \(+\infty\) from the dominated convergence theorem it follows that

\[ |(T(t)f)(x) - f(x)| = |\int_0^t (T(s)Af)(x)ds| \leq t\|Af\|_\infty. \]

Since \( x \in \mathbb{R}^N \) is arbitrary, we conclude that

\[ \|T(t)f - f\|_\infty \leq t\|Af\|_\infty, \]

which proves the proposition. \( \square \)

**Proposition 1.1.6** [2, Proposition, 2.2.9] Let \( f_n \in C_b(\mathbb{R}^N) \) be a bounded sequence of continuous functions converging pointwise to a function \( f \in C_b(\mathbb{R}^N) \) as \( n \) tends to \(+\infty\). Then, \( T(.)f_n \) tends to \( T(.)f \) locally uniformly in \( \mathbb{R}^N \times [0, +\infty) \). Further, if \( f_n \) tends to \( f \) uniformly on compact subsets of \( \mathbb{R}^N \), then \( T(t)f_n \) converges to \( T(t)f \) locally uniformly in \( \mathbb{R}^N \times [0, +\infty) \) as \( n \) tends to \(+\infty\).

**Proof.** To prove the first part of the proposition, we fix \( 0 < T_1 < T_2, R > 0 \), a sequence \( \{f_n\} \subset C_b(\mathbb{R}^N) \) converging pointwise to \( f \in C_b(\mathbb{R}^N) \) and we prove that \( T(.)f_n \) converges to \( T(.)f \) in \([T_1, T_2] \times \overline{B}(0, R)\) as \( n \) tends to \(+\infty\). As it is immediately seen from (1.1.3) and the dominated convergence theorem, \( T(.)f_n \) converges pointwise to \( T(.)f \) in \( \mathbb{R}^N \times (0, +\infty) \) as \( n \) tends to \(+\infty\). Now let \( K > 0 \) be such that \( \sup_{n \in \mathbb{N}} \|f_n\|_\infty \leq K \). Then, taking (1.1.2) into account, we easily deduce that \( \sup_{n \in \mathbb{N}} \|T(t)f_n\|_\infty \leq K \) for any \( t \in [0, T] \). The interior Schauder estimates in Theorem 4.2.7 then imply that the sequence \( \{T(.)f_n\} \) is bounded in \( C^{2+\alpha,1+\alpha/2}(\overline{B}(0, R) \times [T_1, T_2]) \). Hence, by the Ascoli-Arzelà Theorem, there exists a subsequence \( \{T_{n_k}(.)\} \) converging uniformly in \( (\overline{B}(0, R) \times [T_1, T_2]) \) to a function \( v \in C^{2+\alpha,1+\alpha/2}(\overline{B}(0, R) \times [T_1, T_2]) \). Since, \( T(.)f_n \) converges pointwise to \( T(.)f \) in \( \mathbb{R}^N \times (0, +\infty) \), we deduce that \( v = T(.)f \) and the whole sequence \( T(.)f_n \) converges to \( T(.)f \) uniformly in \( (\overline{B}(0, R) \times [T_1, T_2]) \).

Now, we suppose that the sequence \( f_n \subset C_b(\mathbb{R}^N) \) converges uniformly to \( f \) on compact subsets of \( \mathbb{R}^N \) and we show that, for any \( R, T > 0 \), \( T(.)f_n \) tends to \( T(.)f \) uniformly in \( (\overline{B}(0, R) \times [0, T]) \). Possibly replacing \( f_n \) with
\( f_n - f \), we can suppose that \( f \equiv 0 \). Moreover, without loss of generality, we can also assume that \( \sup_{n \in \mathbb{N}} \|f_n\| \leq 1 \). For any \( n \in \mathbb{N} \), let \( \varphi_n \in C_0(\mathbb{R}^N) \) be a nonnegative function such that \( \chi_{B(n-1)} \leq \varphi_n \leq \chi_{B(n)} \). Moreover, for any \( \varepsilon > 0 \), let \( C_{\varepsilon,R} \) be the set defined by

\[
C_{\varepsilon,R} = \{ s \geq 0 : \exists n \in \mathbb{N}, \inf_{(x,t) \in [0,s] \times B(0,R)} (T(t)(\varphi_n - 1))(x) \geq -\varepsilon \}.
\]

Let us prove that \( C_{\varepsilon,R} = [0,\infty) \), for any \( \varepsilon > 0 \). For this purpose, we will show that \( C_{\varepsilon,R} \) is nonempty since it contains 0. To show that \( C_{\varepsilon,R} \) is closed, we fix \( s \in \mathbb{C}_{\varepsilon,R}, s \neq 0 \). Then, there exists a sequence \( \{s_n\} \subset C_{\varepsilon,R} \) converging to \( s \) as \( n \) tends to +\( \infty \). Without loss of generality, we can assume that \( \{s_n\} \) is either decreasing or increasing. Of course, if \( \{s_n\} \) is decreasing, then \( s \in C_{\varepsilon,R} \). So, let us consider the case when \( \{s_n\} \) is increasing. Since \( s_1 \in C_{\varepsilon,R} \), there exists \( n_1 \in \mathbb{N} \) such that

\[
(T(t)(\varphi_n - 1))(x) \geq -\varepsilon, t \in [0,s_1], x \in B(0,R).
\]

Recalling that \( \{\varphi_n\} \) is an increasing sequence, it turns out that \((1.1.9)\) is satisfied by any \( n \geq n_1 \). By the first part of the proof, we know that \( T(.) (\varphi_n - 1) \) converges to 0 uniformly in \( [s_1, s] \times B(0,R) \). Therefore, we can determine \( n_0 \in \mathbb{N} \) such that

\[
(T(t)(\varphi_n - 1))(x) \geq -\varepsilon, t \in [s_1, s], x \in B(0,R), n \geq n_0.
\]

Now, if we take \( \hat{n} = n_0 \vee n_1 \), we deduce that

\[
(T(t)(\varphi_n - 1))(x) \geq -\varepsilon, t \in [0,s], x \in B(0,R).
\]

Hence, \( s \in C_{\varepsilon,R} \). To show that \( C_{\varepsilon,R} \) is open in \([0,\infty)\), we fix \( s \in C_{\varepsilon,R} \) and prove, that for any \( \delta > 0 \), \([s,\delta] \subset C_{\varepsilon,R} \). For this purpose, it suffices to argue as above, observing that \( T(.) (\varphi_n - 1) \) converges to 0, uniformly in \([s,\delta] \times B(0,R) \). Now, since \( p(x, B(m), t) \geq (T(t)\varphi_m)(x) \) for any \( t > 0 \), any \( x \in \mathbb{R}^N \) and any \( m \in \mathbb{N} \), and \( C_{\varepsilon,R} = [0,\infty) \), we easily deduce that, for any arbitrary fixed \( T > 0 \) and any \( R > 0 \), there exists \( m \in \mathbb{N} \) such that

\[
p(x, B(m), t) \geq (T(t)1)(x) - \varepsilon = p(x, \mathbb{R}^N, t) - \varepsilon, t \in [0,T], x \in B(0,R).
\]

Therefore,

\[
|\langle T(t)f_n \rangle(x) - \varepsilon| \leq \int_{B(m)} f_n(y)p(x, dy, t) + \int_{\mathbb{R}^N \setminus B(m)} f_n(y)p(x, dy, t)dy
\]
\[
\leq \sup_{y \in B(m)} |f_n(y)| + p(x, \mathbb{R}^N \setminus B(m))
\leq \sup_{y \in B(m)} |f_n(y)| + \varepsilon,
\]
for any \( t \in [0, T] \) and any \( x \in B(0, R) \). Now the assertion follows. \qed

Let us now prove that \( T(t) \) is irreducible and has the strong Feller property. For this purpose, we recall the following definition.

**Definition 1.1.7** A semigroup \( (T(t))_{t \geq 0} \) in \( B_b(\mathbb{R}^N) \) is irreducible if for any nonempty open set \( U \subset \mathbb{R}^N \) it holds that

\[
(T(t) \chi_U)(x) > 0,
\]
for any \( t > 0 \) and any \( x \in \mathbb{R}^N \). It has the strong Feller property if \( T(t)f \in C_b(\mathbb{R}^N) \) for any \( f \in B_b(\mathbb{R}^N) \).

**Proposition 1.1.8** The semigroup \( (T(t))_{t \geq 0} \) is irreducible and has the strong Feller property.

**Proof.** Showing that the semigroup is irreducible is easy due to the fact that \( p \) is strictly positive (see Theorem 1.1.2). To prove that \( T(t) \) is strong Feller, fix \( f \in B_b(\mathbb{R}^N) \) and let \( \{f_n\} \in C_b(\mathbb{R}^N) \) be a bounded sequence converging pointwise to \( f \) as \( n \) tends to \(+\infty\). Applying the interior Schauder estimates in Theorem 4.2.7 we deduce that for any compact set \( F \subset \mathbb{R}^N \times (0, +\infty) \) there exists a positive constant \( C = C(F) \) such that

\[
\|T(\cdot)f_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(F)} \leq C\|f_n\|_{\infty}, \quad n \in \mathbb{N}.
\]
Since \( \sup_{n \in \mathbb{N}} \|f_n\|_{\infty} \) is finite, and \( (T(t)f_n)(x) \) converges to \( (T(t)f)(x) \) for any \( t \in [0, +\infty) \) and any \( x \in \mathbb{R}^N \), we deduce that \( T(t)f \) is continuous in \( \mathbb{R}^N \) for any \( t > 0 \). \qed

**1.2 The weak generator of \( T(t) \)**

In the previous section we have built a semigroup associated to the given elliptic operator with unbounded coefficients and we have observed that in general it is not strongly continuous in \( C_b(\mathbb{R}^N) \), hence we cannot define it’s
generator in the usual sense. This gap is filled introducing the concept of a weak generator $\hat{A}$ with domain $D(\hat{A}) \subset C_b(\mathbb{R}^N)$.

In [2, chapter 2] the weak generator $(\hat{A}, D(\hat{A}))$ was defined by

$$D(\hat{A}) = \left\{ f \in C_b(\mathbb{R}^N), (x, t) \mapsto \frac{T(t)f(x) - f(x)}{t} \text{ is bounded in } \mathbb{R}^N \times (0, 1) \right\}$$

and for $f \in D(\hat{A})$ it holds

$$\hat{A}f = Af = \lim_{t \to 0} \frac{T(t)f - f}{t} \text{ pointwise.}$$

We have $D(\hat{A}) \subset D_{\max}(A)$ and $D(\hat{A}) = D_{\max}(A)$ if and only if the problem (1.0.1) is uniquely solvable for each $f \in C_b(\mathbb{R}^N)$ in bounded functions. Moreover, $T(\cdot)f(\cdot)$ is for $f \geq 0$ the minimal solution among all positive solutions of the problem (1.0.1).
Chapter 2

Local regularity and integrability of transition kernels

As a first step we recall some local regularity results for the kernel $p$ associated to the minimal semigroup

$$T(t)f(x) = \int_{\mathbb{R}^N} p(x, y, t) f(y) \, dy,$$

actually the minimal among all bounded positive solutions of equation (1.0.1).

2.1 Local regularity of transition kernels

Regularity properties of the kernels $p$ with respect to the variables $(y, t)$ are known even under weaker conditions than our hypothesis (H), see [4]. We combine the results of [4] with the Schauder estimates to obtain regularity of $p$ with respect to all the variables $(x, y, t)$. The proof is similar to the one of Proposition 2.1 in [23].

**Proposition 2.1.1** Under assumption (H) the kernel $p = p(x, y, t)$ is a positive continuous function in $\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$ which enjoys the following properties.

(i) For every $x \in \mathbb{R}^N$, $1 < s < \infty$, the function $p(x, \cdot, \cdot)$ belongs to $\mathcal{H}^{s,1}_{\text{loc}}(\mathbb{R}^N \times (0, \infty))$. In particular, $D_y p \in L^s_{\text{loc}}(\mathbb{R}^N \times (0, \infty))$ and $p(x, \cdot, \cdot)$ is continuous.
(ii) For every $y \in \mathbb{R}^N$ the function $p(\cdot, y, \cdot)$ belongs to $C^{2+\alpha, 1+\alpha/2}_{\text{loc}}(\mathbb{R}^N \times (0, \infty))$ and solves the equation $\partial_t p = A p, \ t > 0$. Moreover

$$\sup_{|y| \leq R} \|p(\cdot, y, \cdot)\|_{C^{2+\alpha, 1+\alpha/2}(B_R \times [\varepsilon, T])} < \infty$$

for every $0 < \varepsilon < T$ and $R > 0$.

(iii) If, in addition, $F \in C^1(\mathbb{R}^N)$, then $p(x, \cdot, \cdot) \in W^{2,1}_{s, \text{loc}}(Q_T)$ for every $x \in \mathbb{R}^N, 1 < s < \infty$, and satisfies the equation $\partial_t p - A^* p = 0$, where $A^* = A_0 - F \cdot D - (V + \text{div} F)$ is the formal adjoint of $A$.

Proof. Assertion (i) is stated in [4, Corollary 3.9].

Let us prove (ii). Since $p(x, \cdot, \cdot)$ is continuous in $(y, t)$ for every fixed $x$, we have $p(x, y, t) < \infty$ for every $t > 0$ and $x, y \in \mathbb{R}^N$. Under this condition, the proof of [25, Theorem 4.4] ensures that $p(\cdot, y, \cdot)$ belongs to $C^{2+\alpha, 1+\alpha/2}_{\text{loc}}(\mathbb{R}^N \times (0, \infty))$ for every $y \in \mathbb{R}^N$ and that $p$ solves $\partial_t p = A p$.

Let us fix $y \in \mathbb{R}^N, 0 < \varepsilon < \tau$, and $t_1 > \tau$. If $|y| \leq R$, then the parabolic Harnack inequality (see, e.g., [17]) yields

$$\sup_{\varepsilon \leq t \leq \tau, x \in B_{2R}} p(x, y, t) \leq C p(0, y, t_1) \leq C p_{|y| \leq R}(0, y, t_1) = M$$

for a suitable $M > 0$. By the interior Schauder estimates (see, e.g., [12, Theorem 8.1.1]) we deduce that

$$\sup_{|y| \leq R} \|p(\cdot, y, \cdot)\|_{C^{2+\alpha, 1+\alpha/2}(B_R \times [\varepsilon, \tau])} \leq C \sup_{|y| \leq R} \|\partial_t p(\cdot, y, \cdot) - A_x p(\cdot, y, \cdot)\|_{C^{\alpha, \frac{\alpha}{2}}(B_{2R} \times [\frac{\varepsilon}{2}, \tau])} + M = CM < \infty.$$  

Finally, we prove that $p$ is continuous in $\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$. If $(x_n, y_n, t_n) \to (x_0, y_0, t_0)$ with $t_0$, then

$$|p(x_n, y_n, t_n) - p(x_0, y_0, t_0)| \leq |p(x_n, y_n, t_n) - p(x_0, y_n, t_0)| + |p(x_0, y_n, t_0) - p(x_0, y_0, t_0)|.$$  

The last term tends to zero by continuity of $p(x_0, \cdot, t_0)$ and the first too, since, by the above estimate, $D_x p$ is uniformly bounded in a neighborhood of $(x_0, y_0, t_0)$. Assertion (iii) follows from standard local parabolic regularity. □
The uniqueness of the bounded solution of (1.0.1) does not hold in general but, it is ensured by the existence of a Lyapunov function [23, Proposition 2.2], that is of a $C^{2+\alpha}_{loc}$-function $W : \mathbb{R}^N \to [0, \infty)$ such that $\lim_{|x| \to \infty} W(x) = +\infty$ and $AW \leq \lambda W$ for some $\lambda > 0$. Lyapunov functions are easily found imposing suitable conditions on the coefficients of $A$. For instance, $W(x) = |x|^2$ is a Lyapunov function for $A$ provided that $\sum_{i} a_{ii}(x) + F(x) \cdot x - |x|^2 V(x) \leq C|x|^2$ for some $C > 0$.

**Proposition 2.1.2** Let $W$ be a Lyapunov function for $A$ and let $u,v \in C_b(\mathbb{R}^N \times [0, T]) \cap C^{2,1}(\mathbb{R}^N \times (0, T))$ solves (1.0.1). Then $u = v$.

**Proof.** It is sufficient to show that if such a function $u$ solves 1.0.1 with $f \geq 0$, then $u \geq 0$. Define $v_\varepsilon = e^{-\lambda t} u + \varepsilon W$, where $\varepsilon > 0$ and $\lambda$ is such that $AW \leq \lambda W$. Then $v_\varepsilon$ has a minimum point $(x_0, t_0) \in \mathbb{R}^N \times [0, T]$. If $v_\varepsilon(x_0, t_0) < 0$, then $t_0 > 0$, since $f \geq 0$, and hence $\partial_t v_\varepsilon(x_0, t_0) \leq 0$. Since $Dv_\varepsilon(x_0, t_0) = 0$ and $\Sigma_{i,j} a_{ij} D_{ij} v_\varepsilon(x_0, t_0) \geq 0$, we have also $(A - \lambda) v_\varepsilon(x_0, t_0) > 0$, and this contradicts the equation $\partial_t v_\varepsilon - (A - \lambda) v_\varepsilon \geq 0$. Therefore, $v_\varepsilon \geq 0$ and, letting $\varepsilon \to 0$, $u \geq 0$.

Now we turn our attention to integrability properties of $p$ and show how they can be deduced from the existence of suitable Lyapunov functions. In the proof of Proposition 2.2.1 below we need to approximate the semigroup $(T(t))_{t \geq 0}$ with semigroups generated by uniformly elliptic operators. This is done in the next lemma.

**Lemma 2.1.3** Assume that $A$ has a Lyapunov function $W$. Take $\eta \in C^\infty_c(\mathbb{R})$ with $\eta(s) = 1$ for $|s| \leq 1$, $\eta(s) = 0$ for $|s| \geq 2$, and define $\eta_n(x) = \eta\left(\frac{|x|}{n}\right)$, $F_n = \eta_n F$, $V_n := \eta_n V$ and $A_n = A_0 + F_n \cdot D - V_n$. Consider the analytic semigroup $(T_n(t))_{t \geq 0}$ generated by $A_n$ in $C_b(\mathbb{R}^N)$. Then, for every $f \in C^{2+\alpha}(\mathbb{R}^N)$ there exists a sequence $(n_k)$ such that $T_{n_k}(\cdot) f(\cdot) \to T(\cdot) f(\cdot)$ in $C^{2,1}(\mathbb{R}^N \times [0, T])$.

**Proof.** Let $u_n(x, t) = T_n(t) f(x)$, $u(x, t) = T(t) f(x)$ and fix a radius $\varrho > 0$. If $n > \varrho + 1$ the Schauder estimates for the operator $A$ (see e.g. [12, Theorem 8.1.1]) yield

$$
\|u_n\|_{C^{2+\alpha, 1+\alpha/2}(B_{\varrho} \times [0, T])} \leq C_{\varrho} \|f\|_{C^{2+\alpha}(\mathbb{R}^N)}.
$$

By a standard diagonal argument we find a subsequence $(n_k)$ such that $u_{n_k}$ converges to a function $u$ in $C^{2,1}(\mathbb{R}^N \times (0, \infty))$. Since $\partial_t u_{n_k} - A u_{n_k} = 0$ in $B_{\varrho} \times [0, T]$ for $n_k > \varrho$ we have $\partial_t u - A u = 0$ in $\mathbb{R}^N \times [0, T]$. Moreover,
u(x,0) = f(x) and |u(x,t)| \leq \|f\|_{\infty}, since this is true for \(u_n\). By Proposition 2.1.2 we infer that \(u(x,t) = T(t)f(x)\).

\section{2.2 Integrability of transition kernels}

The integrability of Lyapunov functions with respect to the measures \(p(x,y,t)\) is given by the following result, which is an extension of [26, Lemma 3.9], where the case \(V = 0\) is considered.

\begin{proposition}
A Lyapunov function \(W\) is integrable with respect to the measures \(p(x,\cdot,t)\). Setting
\[
\zeta(x,t) = \int_{\mathbb{R}^N} p(x,y,t)W(y)\,dy,
\]
the inequality \(\zeta(x,t) \leq e^{\lambda t}W(x)\) holds. Moreover, \(|AW|\) is integrable with respect to \(p(x,\cdot,t)\), \(\zeta \in C^{2,1}(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty))\) and \(D_t\zeta(x,t) \leq \int_{\mathbb{R}^N} p(x,y,t)AW(y)\,dy\).
\end{proposition}

\begin{proof}
For \(\alpha \geq 0\), set \(W_{\alpha} := W \wedge \alpha\) and \(\zeta_{\alpha}(x,t) := \int_{\mathbb{R}^N} p(x,y,t)W_{\alpha}(y)\,dy\).

Let us consider, for every \(0 < \varepsilon < 1\), \(\psi_{\varepsilon} \in C^{\infty}(\mathbb{R})\) such that \(\psi_{\varepsilon}(t) = t\) for \(t \leq \alpha\), \(\psi_{\varepsilon}\) constant in \([\alpha + \varepsilon,\infty)\), \(\psi_{\varepsilon}' \geq 0\), and \(\psi_{\varepsilon}'' \leq 0\). Since \(\psi_{\varepsilon}'' \leq 0\) one deduces that
\[
t\psi_{\varepsilon}'(t) \leq \psi_{\varepsilon}(t), \quad \forall t \geq 0.
\]

Now we approximate \(A\) and \((T(t))_{t \geq 0}\) with \(A_n := A_0 + F_n \cdot \nabla - V_n\) and \((T_n(t))_{t \geq 0}\) as in Lemma 2.1.3. Denoting by \(p_n(x,y,t)\) the kernel of \((T_n(t))_{t \geq 0}\), since \(\psi_{\varepsilon} \circ W \in C_b^{2+\alpha}(\mathbb{R}^N)\) we have
\[
\partial_t T_n(t)(\psi_{\varepsilon} \circ W)(x) = \int_{\mathbb{R}^N} p_n(x,y,t)A_n(\psi_{\varepsilon} \circ W)(y)\,dy.
\]

On the other hand, by (2.2.2), we obtain
\[
A_n(\psi_{\varepsilon} \circ W)(x) = \psi_{\varepsilon}'(W(x))A_nW(x) + V_n(x) [\psi_{\varepsilon}'(W(x))W(x) - \psi_{\varepsilon}(W(x))] \\
+ \psi_{\varepsilon}''(W(x)) \sum_{i,j=1}^{N} a_{ij}(x) D_iW(x)D_jW(x) \\
\leq \psi_{\varepsilon}'(W(x))A_nW(x).
\]
Thus,
\[ \partial_t T_n(t)(\psi \circ W)(x) \leq \int_{\mathbb{R}^N} p_n(x, y, t)\psi'(W(y))A_nW(y) \, dy \]
and also
\[ \partial_t T_n(t)(\psi \circ W)(x) \leq \int_{\mathbb{R}^N} p_n(x, y, t)\psi'(W(y))AW(y) \, dy, \]
if \( n \) is sufficiently large since, for fixed \( \varepsilon \), the function \( \psi'(W(y)) \) has compact support. Letting \( n \to \infty \) and using Lemma 2.1.3 (possibly passing to a subsequence) we deduce
\[ \partial_t T(t)(\psi \circ W)(x) \leq \int_{\mathbb{R}^N} p(x, y, t)\psi'(W(y))AW(y) \, dy. \] (2.2.3)
Next we observe that \( \psi \circ W \leq \alpha + 1, \psi'(t) \to \chi_{(0, \alpha]}(t) \), and \( \psi \circ W \to W_\alpha \) pointwise as \( \varepsilon \to 0 \). From [2, Proposition 2.2.9] we deduce that \( T(t)(\psi \circ W) \to T(t)W_\alpha \) in \( C^{1,1}(\mathbb{R}^N \times (0, \infty)) \). So, letting \( \varepsilon \to 0 \) in (2.2.3) and using dominated convergence in the right hand side (all the integrals can be taken on the compact set \( \{W \leq \alpha + 1\} \), where \( AW \) is bounded) we get
\[ D_t \zeta_\alpha(x, t) \leq \int_{\{W \leq \alpha\}} p(x, y, t)AW(y) \, dy. \] (2.2.4)
To conclude we proceed as in the proof of [26, Lemma 3.9]. From (2.2.4) we obtain
\[ D_t \zeta_\alpha(x, t) \leq \lambda \zeta_\alpha(x, t) \] (2.2.5)
and hence, by Gronwall’s lemma, \( \zeta_\alpha(x, t) \leq e^{\lambda t}W_\alpha(x) \). Letting \( \alpha \to \infty \) we obtain \( \zeta(x, t) \leq e^{\lambda t}W(x) \) and then \( W \) is summable with respect to the measure \( p(x, \cdot, t) \). The inequality \( 0 \leq \zeta_\alpha \leq \zeta \) and the interior Schauder estimates show that the family \( \{\zeta_\alpha\} \) is relatively compact in \( C^{1,1}(\mathbb{R}^N \times (0, \infty)) \). Since \( \zeta_\alpha \to \zeta \) pointwise as \( \alpha \to +\infty \), it follows that \( \zeta \in C^{1,1}(\mathbb{R}^N \times (0, \infty)) \). Moreover, the inequality \( \zeta_\alpha(x, t) \leq \zeta(x, t) \leq e^{\lambda t}W(x) \) implies that \( \zeta(\cdot, t) \to W(\cdot) \) as \( t \to 0^+ \), uniformly on compact sets. Set \( E = \{x \in \mathbb{R}^N : AW(x) \geq 0\} \). Clearly
\[ \int_E p(x, y, t)AW(y) \, dy \leq \lambda \int_E p(x, y, t)W(y) \, dy \leq \lambda \zeta(x, t) < \infty. \] (2.2.6)
Moreover, letting \( \alpha \to +\infty \) in (2.2.3), we obtain that
\[ D_t \zeta(x, t) \leq \liminf_{\alpha \to +\infty} \int_{\{W \leq \alpha\}} p(x, y, t)AW(y) \, dy. \]
This fact and (2.2.6) imply that \( |AW| \) is summable with respect to \( p(x, \cdot, t) \) and that the above \( \liminf \) is a limit, so that the proof is complete. \( \square \)
Assuming that $AW$ tends to $-\infty$ faster than $-W$ one obtains, by Proposition 2.2.1, that the function $\zeta$ in (2.2.1) is bounded with respect to the space variables, see [26, Theorem 3.10] for the case $V = 0$.

**Proposition 2.2.2** Assume that the Lyapunov function $W$ satisfies the inequality $AW \leq -g(W)$ where $g : [0, \infty) \to \mathbb{R}$ is a differentiable convex function such that $g(0) \leq 0$, $\lim_{s \to +\infty} g(s) = +\infty$ and $1/g$ is integrable in a neighbourhood of $+\infty$. Then for every $a > 0$ the function $\zeta$ defined in (2.2.1) is bounded in $\mathbb{R}^N \times [a, \infty)$. Moreover, the semigroup $(T(t))_{t \geq 0}$ is compact in $C_b(\mathbb{R}^N)$.

**Proof.** Observe that $g(s) \leq sg'(s)$, since $g$ is convex with $g(0) \leq 0$. Let us prove that

$$\int_{\mathbb{R}^N} p(x, y, t)g(W(y)) \, dy \geq g(\zeta(x, t)). \tag{2.2.7}$$

For, fix $x$ and $t$ and set $s_0 = \zeta(x, t)$. Then, for all $y \in \mathbb{R}^N$ we have

$$g(W(y)) \geq g(s_0) + g'(s_0)(W(y) - s_0)$$

and therefore, multiplying by $p(x, y, t)$ and integrating

$$\int_{\mathbb{R}^N} p(x, y, t)g(W(y)) \, dy \geq g(s_0)\int_{\mathbb{R}^N} p(x, y, t) \, dy + g'(s_0)s_0(1 - \int_{\mathbb{R}^N} p(x, y, t) \, dy) \geq g(s_0).$$

From Proposition 2.2.1 and (2.2.7) we deduce

$$D_t \zeta(x, t) \leq \int_{\mathbb{R}^N} p(x, y, t)AW(y) \, dy \leq -\int_{\mathbb{R}^N} p(x, y, t)g(W(y)) \, dy \leq -g(\zeta(x, t))$$

and therefore $\zeta(x, t) \leq z(x, t)$, where $z$ is the solution of the ordinary Cauchy problem

$$\begin{cases}
  z' = -g(z) \\
  z(x, 0) = W(x).
\end{cases}$$

Let $\ell$ denote the greatest zero of $g$. Then $z(x, t) \leq \ell$ if $W(x) \leq \ell$. On the other hand, if $W(x) > \ell$, then $z$ is decreasing and satisfies

$$t = \int_{z(x, t)}^{W(x)} \frac{ds}{g(s)} \leq \int_{z(x, t)}^{\infty} \frac{ds}{g(s)}. \tag{2.2.8}$$

This inequality easily yields, for every $a > 0$, a constant $C(a)$ such that $z(x, t) \leq C(a)$ for every $t \geq a$ and $x \in \mathbb{R}^N$.

The compactness of $(T(t))_{t \geq 0}$ in $C_b(\mathbb{R}^N)$ can be proved as in [26, Theorem 3.10].
Remark 2.2.3 If \( \int_{\mathbb{R}^N} p(x, y, t) \, dy = 1 \) (as is the case if \( V = 0 \)) then (2.2.7) follows from Jensen’s inequality and the condition \( g(0) \leq 0 \) is not needed.

Let us state a condition under which certain exponentials or polynomials are Lyapunov functions. Using the same procedure as for the case \( V = 0 \) (see [23, Proposition 2.5 and 2.6]) we obtain the following results.

**Proposition 2.2.4** Let \( \Lambda \) be the maximum eigenvalue of \((a_{ij})\) as in (H).

Assume that

\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta \beta |x|^{\beta-1}} \right) < -c,
\]

for some \( c, \delta > 0, \beta > 1 \) such that \( \delta < (\beta \Lambda)^{-1} \). Then \( W(x) = \exp\{\delta |x|^\beta\} \) is a Lyapunov function. Moreover, if \( \beta > 2 \), there exist positive constants \( c_1, c_2 \) such that

\[
\zeta(x, t) \leq c_1 \exp\left( c_2 t^{-\beta/(\beta-2)} \right)
\]

for \( x \in \mathbb{R}^N, \ t > 0 \).

**Proof.** Let \( W(x) = \exp\{\delta |x|^\beta\} \) and set \( G_i = F_i + \sum_j D_j a_{ij} \). We obtain, by a straightforward computation,

\[
AW(x) = \delta \beta |x|^{\beta-1} e^\delta |x|^\beta \left( \sum_i a_i(x) + \frac{\beta - 2}{|x|^3} \sum_{ij} a_{ij}(x) x_i x_j \right.
\]

\[
+ \delta \beta |x|^{\beta-3} \sum_{ij} a_{ij}(x) x_i x_j + G \cdot \frac{x}{|x|} - \frac{V(x)}{\delta \beta |x|^{\beta-1}} \left. \right)
\]

\[
\leq C_1 |x|^{\beta-1} e^\delta |x|^\beta \left( 1 + (\delta \beta \Lambda - c)|x|^{\beta-1} \right)
\]

\[
\leq -C_2 |x|^{2\beta-2} e^\delta |x|^\beta \leq 0
\]

for \( |x| \) large. This shows that \( W \) is a Lyapunov function. Finally, if \( \beta > 2 \) it follows that \( AW \leq -g(W) \) with \( g(s) = C_3 s (\log s)^{2-2/\beta} - C_4 \), for suitable \( C_3, C_4 > 0 \). Then Proposition 2.2.2 yields the boundedness of \( \zeta(\cdot, t) \). To obtain (2.2.10) we recall that \( \zeta \leq z \) where \( z \) satisfies (2.2.8). If \( \ell \) denotes the zero of \( g \) and \( z(x, t) \leq 2\ell \) we have simply to choose a suitable \( c_1 \). If \( z(x, t) \geq 2\ell \), then

\[
t \leq \int_{z}^{\infty} \frac{ds}{g(s)} \leq C_5 \int_{z}^{\infty} \frac{ds}{s (\log s)^{2-2/\beta}} \leq C_6 (\log z)^{2/\beta-1}
\]

and (2.2.10) follows. \( \square \)
The right hand side of (2.2.10) becomes very big as \( t \to 0 \). In order to have a milder behaviour we investigate when polynomials are Lyapunov functions.

**Proposition 2.2.5** Assume that

\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{|x|}{2\alpha} V(x) \right) < 0, \tag{2.2.11}
\]

for some \( \alpha > 0, \beta > 2 \). Then \( W(x) = (1 + |x|^2)^\alpha \) is a Lyapunov function and there exists a positive constant \( c \) such that

\[
\zeta(x, t) \leq ct^{-(2\alpha)/\beta - 2} \tag{2.2.12}
\]

for \( x \in \mathbb{R}^N, \ 0 < t \leq 1 \).

**Proof.** We have, with the notation of Proposition 2.2.4,

\[
AW(x) = \left( 1 + |x|^2 \right)^\alpha \left( \frac{2\alpha}{1 + |x|^2} \sum_i a_{ii}(x) + \frac{4\alpha(\alpha - 1)}{(1 + |x|^2)^2} \sum_{i,j} a_{ij}(x)x_i x_j \right. \\
+ \frac{2\alpha}{1 + |x|^2} G \cdot x - V(x) \left. \right) \\
\leq -C_1 \left( 1 + |x|^2 \right)^{\alpha+(\beta-2)/2} = -C_1 W^\gamma
\]

for \( |x| \) large and with \( \gamma = 1 + (\beta - 2)/(2\alpha) \) \( > 1 \). This shows \( AW \leq -g(W) \) with \( g(s) = C_2 s^\gamma - C_3 \) for suitable \( C_2, C_3 > 0 \). Proceeding as in the proof of (2.2.10) one shows (2.2.12), the only difference being that the function \( t^{-(2\alpha)/\beta - 2} \) goes to 0 as \( t \to +\infty \), and then the estimate is not true, in general, for all \( t > 0 \). \( \square \)

**Remark 2.2.6** Proposition 2.2.4 will be used to check the integrability of \( |F|^k \) and \( V^k \) with respect to \( p \), assuming that \( |F|, V \) grow at infinity not faster than \( \exp\{|x|^{\gamma}\} \) for some \( \gamma < \beta \).

**Remark 2.2.7** Conditions (2.2.9), (2.2.11) are assumptions on the radial component of \( F \) and, if \( 0 < c < \infty \), say that the inward radial component of \( F \) has a prescribed polynomial behaviour. Of course, changing \( x/|x| \) to \( (x - x_0)/|x - x_0| \) leads to new conditions that, though not equivalent to (2.2.9), (2.2.11), yield similar conclusions.
Finally, we clarify in which sense the identity $\partial_t p = A^*_yp$ is satisfied.

**Lemma 2.2.8** Let $0 \leq a < b$ and $\varphi \in C^{2,1}_c(Q(a,b))$. Then

$$\int_{Q(a,b)} \left( \partial_t \varphi(y,t) + A\varphi(y,t) p(x,y,t) \right) dy dt = \int_{\mathbb{R}^N} \left( p(x,y,b) \varphi(y,b) - p(x,y,a) \varphi(y,a) \right) dy. \tag{2.2.13}$$

**Proof.** If $\psi \in C^{2,1}_c(Q(a,b))$, then $\partial_t T(t)\psi = T(t)A\psi$. If $\varphi \in C^{2,1}_c(Q(a,b))$, then

$$\partial_t (T(t)\varphi(.,t)) = T(t)\partial_t \varphi(.,t) + T(t)A\varphi(.,t).$$

Integrating this identity over $[a,b]$ and writing $T(t)$ in terms of the kernel $p$, we obtain (2.2.13). $\square$
Chapter 3

Uniform and pointwise bounds on transition kernels

In this chapter we fix $T > 0$ and consider $p$ as a function of $(y, t) \in \mathbb{R}^N \times (0, T)$ for arbitrary, but fixed, $x \in \mathbb{R}^N$. Further, fix $0 < a_0 < a < b < b_0 \leq T$ and assume for definiteness $b_0 - b \geq a - a_0$. Setting

$$
\Gamma(k, x, a_0, b_0) := \left( \int_{Q(a_0, b_0)} (1 + |F(y)|^k + V(y)^k)p(x, y, t) \, dy \, dt \right)^{\frac{1}{k}}. \quad (3.0.1)
$$

The proofs of Proposition 3.1, Lemma 3.1 and Proposition 3.2 in [23] remain valid for the case $V \neq 0$. So, we obtain that

$$p \in H^{s,1}(Q(a, b)) \quad \text{for all } s \in (1, k),$$

provided that $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N + 2$. Hence, by the embedding theorem for $H^{s,1}$, $s > N + 2$, (see [23, Theorem 7.1]), we have

**Theorem 3.0.9** If $\Gamma(k, x, a_0, b_0) < \infty$ for some $k > N + 2$, then $p$ belongs to $L^\infty(Q(a, b))$.

### 3.1 Uniform bounds on transition densities

To obtain uniform and pointwise bounds on $p$ we introduce the functions

$$
\Gamma_1(k, x, a_0, b_0) = \left( \int_{Q(a_0, b_0)} (1 + |F(y)|^k)p(x, y, t) \, dy \, dt \right)^{\frac{1}{k}}, \quad (3.1.1)
$$
\begin{align*}
\Gamma_2(k, x, a_0, b_0) &= \left( \int_{Q(a_0, b_0)} V^\frac{k}{2}(y)p(x, y, t) \, dy \, dt \right)^\frac{2}{k}. \quad (3.1.2)
\end{align*}

Clearly \( \Gamma_1(k, x, a_0, b_0) + \Gamma_2(k, x, a_0, b_0) \leq C T(k, x, a_0, b_0) \).

The following result shows that only the assumption \( \Gamma_1(k, x, a_0, b_0), \Gamma_2(k, x, a_0, b_0) < \infty \) for some \( k > N + 2 \) is needed to obtain the boundedness of \( p \).

**Theorem 3.1.1** If \( \Gamma_1(k, x, a_0, b_0), \Gamma_2(k, x, a_0, b_0) < \infty \) for some \( k > N + 2 \) then

\[ \|p\|_{L^\infty(Q(a,b))} \leq C \left( \Gamma_1^k(k, x, a_0, b_0) + \Gamma_2^k(k, x, a_0, b_0) + \frac{b_0 - a_0}{(a - a_0)^2} \right). \quad (3.1.3) \]

**Proof.** Step 1. Assume first that \( \Gamma(k, x, a_0, b_0) < \infty \) so that \( p \in L^\infty(Q(a,b)) \) for every \( a_0 < a < b < b_0 \), by Theorem 3.0.9 and consider \( q \equiv \eta^\frac{k}{2}p \in L^\infty(Q_T) \) where \( \eta \) is a smooth function with compact support in \((a_0, b_0)\) such that \( 0 \leq \eta \leq 1, \eta(t) = 1 \) for \( a \leq t \leq b \). Clearly \( q \in L^\infty(Q_T) \).

Let \( \varphi \in C^{2,1}(Q_T) \) be such that \( \varphi(\cdot, t) \) has compact support for every \( t \).

From (2.2.13) we obtain

\[ \left| \int_{Q_T} q(\partial_t \varphi + A_0 \varphi) \, dy \, dt \right| = \left| \int_{Q_T} (qF \cdot D\varphi - Vq \varphi + k - 2 p \varphi \eta q^{-\frac{k-2}{2}} \partial_t \eta) \, dy \, dt \right|. \]

Next we note that

\[ \|p\eta^{-\frac{k-2}{2}}\|_{L^k(Q_T)} \leq \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0 - a_0)^{\frac{k}{k-2}}. \]

and that

\[ \|Fq\|_{L^k(Q_T)} \leq \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \Gamma_1(k, x, a_0, b_0), \quad \|Vq\|_{L^k(Q_T)} \leq \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \Gamma_2(k, x, a_0, b_0). \]

Since also

\[ \|q\|_{L^k(Q_T)} \leq \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}}(b_0 - a_0)^{\frac{1}{k}}, \quad \|q\|_{L^k(Q_T)} \leq \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}}(b_0 - a_0)^{\frac{2}{k}}, \]

Theorem 3.0.9 now implies that

\[ \|q\|_{L^\infty(Q_T)} \leq C \left( \|q\|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \Gamma_1(k, x, a_0, b_0) + \|q\|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left( \Gamma_2(k, x, a_0, b_0) + \frac{(b_0 - a_0)^{\frac{2}{k}}}{a - a_0} \right) \right). \]
and hence, after a simple calculation,
\[ \|q\|_{L^\infty(Q_T)} \leq C \left( \Gamma_1^h(k, x, a_0, b_0) + \Gamma_2^h(k, x, a_0, b_0) + \frac{b_0 - a_0}{(a - a_0)^{\frac{h}{2}}} \right) \]
and (3.1.3) follows.

**Step 2.** Let us now consider the general case. Fix a smooth function \( \theta \in C_\infty^\infty(\mathbb{R}) \) such that \( \theta(s) = 1 \) for \( |s| \leq 1 \), \( \theta(s) = 0 \) for \( |s| \geq 2 \) and define \( \theta_n(x) = \theta(|x|/n), V_n = V \theta_n \). We consider the minimal semigroup \((U_n(t))_{t \geq 0}\) generated in \( C_b(\mathbb{R}^N) \) by the operator \( A_n = A_0 + F \cdot D - V_n \). Since \( V_n \leq V \) the procedure for constructing the minimal semigroup recalled in Section 2 and the maximum principle yields \( U_n(t)f \leq T(t)f \) for every \( f \in C_b(\mathbb{R}^N) \). If \( p_n \) denotes the kernel of \( U_n \) the above inequality is equivalent to \( p_n(x, y, t) \leq p(x, y, t) \).

To show that \( p_n \) converges pointwise to \( p \) we consider the analytic semigroup \((T_n(t))_{t \geq 0}\) generated by \( A \) on \( C_b(B_n) \), under Dirichlet boundary conditions (\( B_n \) is the ball of centre 0 and radious \( n \)). Since \( V_n = V \) in \( B_n \), the maximum principle gives \( T_n(t)f \leq U_n(t)f \leq T(t)f \) in \( B_n \) for every \( f \in C_b(\mathbb{R}^N), f \geq 0 \). Then \( r_n(x, y, t) \leq p_n(x, y, t) \leq p(x, y, t) \) for \( x, y \in B_\rho \) with \( \rho < n \), where \( r_n \) is the kernel of \( T_n \) in \( B_n \). Letting \( n \to \infty \) we see that \( p_n \to p \) pointwise, since this is true for \( r_n \), see [25, Theorem 4.4].

The proof now easily follows by approximation from Step 1. Let \( \Gamma_i^n(k, x, a_0, b_0) \) be the functions defined in (3.0.1), (3.1.1), (3.1.2) relative to \( p_n \). Since \( p_n \leq p \) and \( V_n \leq V \) it follows that \( \Gamma_i^n(k, x, a_0, b_0) \leq \Gamma_i(k, x, a_0, b_0) \). Moreover, \( \Gamma_i^n(k, x, a_0, b_0) < \infty \) for every \( n \), since \( V_n \) is bounded. Then we obtain from Step 1
\[ \|p_n\|_{L^\infty(Q(a,b))} \leq C \left( \Gamma_1^h(k, x, a_0, b_0) + \Gamma_2^h(k, x, a_0, b_0) + \frac{b_0 - a_0}{(a - a_0)^{\frac{h}{2}}} \right) \]
and the statement follows letting \( n \to \infty \).

\( \square \)

### 3.2 Pointwise bounds on transition kernels

Now we apply similar techniques to obtain pointwise bounds.

We consider the following assumption depending on the weight function \( \omega \) which, in our examples, will be a polynomial or an exponential.

**(H1)** \( W_1, W_2 \) are Lyapunov functions for \( A \), \( W_1 \leq W_2 \) and there exists \( 1 \leq \omega \in C^2(\mathbb{R}^N) \) such that
(i) \( \omega \leq cW_1, \ |D\omega| \leq c\omega^{\frac{1}{k-1}}W_1^\frac{1}{k}, \ |D^2\omega| \leq c\omega^{\frac{k-2}{k}}W_1^\frac{2}{k} \)

(ii) \( \omega V^\frac{1}{k} \leq cW_2 \) and \( |F|^k \leq cW_2 \)

for some \( k > N + 2 \) and a constant \( c > 0 \).

We denote by \( \zeta_1, \zeta_2 \) the functions defined by (2.2.1) and associated with \( W_1, W_2 \), respectively.

By Proposition 2.2.1 we know that (H1) implies \( \Gamma_i(k, x, a_0, b_0) < \infty \) for \( i = 1, 2 \). In particular, since \( k > N + 2 \), Theorem 3.1.1 shows that \( p(x, ' , . ) \in L^\infty(Q(a, b)) \) for every \( x \in \mathbb{R}^N \).

The use of different Lyapunov functions allows us to obtain more precise estimates in the theorem below and its corollaries.

The proof of the following result is similar to the one of [23, Theorem 4.1]. For reader’s convenience we give the details of the proof.

**Theorem 3.2.1** Assume (H1). Then, there exists a constant \( C > 0 \) such that

\[
0 < \omega(y)p(x, y, t) \leq C \left( \int_{a_0}^{b_0} \zeta_2(x, t) \, dt + \frac{1}{(a - a_0)^2} \int_{a_0}^{b_0} \zeta_1(x, t) \, dt \right) \tag{3.2.1}
\]

for all \( x, y \in \mathbb{R}^N, a \leq t \leq b \).

**Proof.** **Step 1.** Assume first that \( \omega \) is bounded. As in the proof of Theorem 3.1.1 we choose a smooth function \( \eta(t) \) such that \( \eta(t) = 1 \) for \( a \leq t \leq b \) and \( \eta(t) = 0 \) for \( t \leq a_0 \) and \( t \geq b_0 \), \( 0 \leq \eta' \leq \frac{2}{a-a_0} \). We consider \( \psi \in C^{2,1}(Q_T) \) such that \( \psi(\cdot, T) = 0 \) and such that \( \psi(\cdot, t) \) has compact support for all \( t \). Setting \( q = \eta^k p \) and taking \( \varphi(y, t) = \omega(y)\psi(y, t) \), from (2.2.13) we obtain

\[
\int_{Q_T} \omega q \left( -\partial t \psi - A_0 \psi \right) \, dy \, dt = \int_{Q_T} \left[ q \left( \psi A_0 \omega + 2 \sum_{i,j=1}^N a_{ij} D_i \omega D_j \psi + \omega F \cdot D \psi + \psi F \cdot D \omega - V \omega \psi \right) \right] + \frac{k}{2} \omega \psi \tag{3.2.2}
\]

Since \( \omega \) is bounded, then \( \omega q \in L^1(Q_T) \cap L^\infty(Q_T) \), by Theorem 3.1.1 and then Theorem 3.0.9 yields

\[
\| \omega q \|_{L^\infty(Q_T)} \leq C \left( \| \omega q \|_{L^1(Q_T)} + \| \omega q \|_{L^2(Q_T)} + \| q D^2 \omega \|_{L^2(Q_T)} + \| q D \omega \|_{L^2(Q_T)} \right.
\]

\[+ \| \omega q F \|_{L^1(Q_T)} + \| q F D \omega \|_{L^2(Q_T)} + \| q V \omega \|_{L^2(Q_T)} + \frac{1}{a-a_0} \| p \omega \eta^\frac{k-2}{2} \|_{L^2(Q_T)} \right).
\]
Next observe that
\[
\|\omega q\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \|\omega q\|_{L^1(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{1}{k}},
\]
\[
\|\omega q\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \|\omega q\|_{L^1(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{2}{k}},
\]
and that, by (H1)(ii),
\[
\|\omega q F\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \|\omega q F\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^{\frac{1}{k}},
\]
\[
\|\omega q V\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \|\omega q V\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^{\frac{2}{k}}.
\]
Moreover, as in the proof of Theorem 3.1.1 one has
\[
\|\omega p \eta \|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \|\omega p\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{2}{k}}.
\]
Next we combine (H1)(i) and (H1)(ii) to estimate the remaining terms
\[
\|D\omega q F\|_{L^k(Q_T)} \leq \left( \int_{Q_T} q \bar{\omega} \|\omega q F\|_{L^k(Q_T)} \right)^{\frac{2}{k}} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^{\frac{2}{k}}
\]
and, similarly,
\[
\|D^2\omega q\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{2}{k}}
\]
\[
\|D\omega q\|_{L^k(Q_T)} \leq \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{1}{k}}.
\]
Collecting similar terms and recalling that $W_1 \leq W_2$ we obtain
\[
\|\omega q\|_{L^k(Q_T)} \leq C \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^{\frac{1}{k}}
\]
\[
\|\omega q\|_{L^k(Q_T)} \leq C \|\omega q\|_{L^k(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^{\frac{2}{k}} + \frac{1}{a-a_0} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^{\frac{1}{k}}
\]
36
hence, after simple computations,
\[ \|\omega q\|_{L^\infty(Q_T)} \leq C \left( \int_{a_0}^{b_0} \zeta_2 \, dt + \frac{1}{(a-a_0)^\beta} \int_{a_0}^{b_0} \zeta_1 \, dt \right) . \]

and (3.2.1) follows for a bounded \( \omega \)

**Step 2.** If \( \omega \) is not bounded, we consider \( \omega \varepsilon = \omega / (1 + \varepsilon \omega) \). A straightforward computation shows that \( \omega \varepsilon \) satisfies \( (H1) \) with a constant \( C \) independent of \( \varepsilon \). Therefore, from Step 1 we obtain
\[
0 < \omega \varepsilon (y)p(x,y,t) \leq C \left( \int_{a_0}^{b_0} \zeta_2(x,t) \, dt + \frac{1}{(a-a_0)^\beta} \int_{a_0}^{b_0} \zeta_1(x,t) \, dt \right) (3.2.3)
\]
with \( C \) independent of \( \varepsilon \) and, letting \( \varepsilon \to 0 \) the statement is proved. \( \Box \)

**Corollary 3.2.2** Assume that
\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta \beta |x|^{\beta-1}} \right) < -c, \quad 0 < c < \infty \quad (3.2.4)
\]
for some \( \delta > 0, c > 0, \beta > 2 \) such that \( \delta < (\beta \Lambda)^{-1}c \), where \( \Lambda \) is the maximum eigenvalue of \( (a_{ij}) \), and that \( V(x) + |F(x)| \leq c_1 e^{c_2|x|^{\beta-\epsilon}} \) for some \( \epsilon, c_1, c_2 > 0 \). Then
\[
0 < p(x,y,t) \leq c_3 \exp \left( c_4 t^{-\frac{\beta}{\beta-2}} \right) \exp \left( -\delta |y|^2 \right)
\]
for \( x, y \in \mathbb{R}^N, 0 < t \leq T \), for suitable \( c_3, c_4 > 0 \).

**Proof.** We take \( \omega(y) = e^{\delta |y|^\beta}, W_1(y) = W_2(y) = e^{\gamma |y|^\beta} \) for \( \delta < \gamma < (\beta \Lambda)^{-1}c \) and use Theorem 3.2.1 with \( a = t \) and \( a - a_0 = b_0 - b = b - a = (1/2)t \). The thesis then follows using Proposition 2.2.4. \( \Box \)

**Example 3.2.3** (i) The above corollary applies with any \( \gamma < (\beta \Lambda)^{-1}c \) and without any restriction on \( V \geq 0 \) when
\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} \right) < -c, \quad 0 < c < \infty
\]
for some \( \beta > 2 \) and \( |F(x)| \leq c_1 e^{c_2|x|^{\beta-\epsilon}} \) for some \( \epsilon, c_1, c_2 > 0 \). This is obvious if \( V = 0 \) and, in the general case, it follows by observing that the kernel \( p \) is pointwise dominated, by the maximum principle, by the corresponding kernel of the operator with \( V = 0 \).
(ii) Let us consider the Schrödinger operator \( \Delta - a^2|x|^s \) with \( a > 0, s > 2 \). Then Corollary 4.2.2 applies with \( \beta = 1 + s/2 \) and any \( \delta < 2a/(s+2) \). This yields

\[
0 < p(x, y, t) \leq c_3 \exp \left( c_4 t^{-\frac{s+2}{s}} \right) \exp \left( -\delta |y|^\frac{s+2}{2} \right) := c(t)\phi(y).
\]

Using the symmetry of \( p \) and the semigroup law (see [22, Example 3.13]), we obtain

\[
p(x, y, t) \leq c_3 \exp \left( c_4 t^{-\frac{s+2}{s}} \right) \exp \left( -\delta |x|^\frac{s+2}{2} \right) \exp \left( -\delta |y|^\frac{s+2}{2} \right).
\]

This estimate was obtained in [22, Example 3.13].

(iii) Let us generalize the previous situation to the case of the operators

\[
A = \Delta - |x|^r x |x| \cdot D - |x|^s
\]

with \( r > 1 \). We distinguish between three cases.

(a) If \( s < 2r \), then \( \beta = r + 1 \) and \( \delta \) can be any positive number less than \( 1/(r+1) \). Therefore

\[
0 < p(x, y, t) \leq c_1 \exp \left( c_2 t^{-\frac{r+1}{r}} \right) \exp \left( -\delta |y|^{r+1} \right).
\]

(b) If \( s = 2r \), then \( \beta = r + 1 \) as before but now \( \delta \) must be less than \( (1 + \sqrt{5})/2(r + 1) \).

(c) If \( s > 2r \), then \( \beta = 1 + s/2 \) and \( \delta < 2/(s+2) \). Then we get, as in (ii)

\[
0 < p(x, y, t) \leq c_1 \exp \left( c_2 t^{-\frac{s+2}{s}} \right) \exp \left( -\delta |x|^\frac{s+2}{2} \right) := c(t)\phi(y).
\]

In this case one can also obtain estimates with respect to \( x \) proceeding as in (ii). We consider the formal adjoint \( A^* = \Delta + |x|^r \frac{x}{|x|} \cdot D + (N + r - 1)|x|^{-1} - |x|^s \). The associated minimal semigroup has the kernel \( p^*(x, y, t) = p(y, x, t) \) which satisfies (3.2.5), by the same argument as above. This yields \( p(t, x, y) \leq c(t)\phi(x) \) and, proceeding as in (ii),

\[
p(x, y, t) \leq c_1 \exp \left( c_2 t^{-\frac{s+2}{s}} \right) \exp \left( -\delta |x|^\frac{s+2}{2} \right) \exp \left( -\delta |y|^\frac{s+2}{2} \right).
\]
Under conditions similar to those of Corollary 3.2.2, the estimate of \( p \) can be improved with respect to the time variable, loosing the exponential decay in \( y \).

**Corollary 3.2.4** Assume that

\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{|x|}{2\alpha} V(x) \right) < 0,
\]

for some \( \alpha > 0 \) and \( \beta > 2 \). If \( |F(x)| + \sqrt{V(x)} \leq c(1 + |x|^2)^{\gamma_1} \) and \( \omega(x) := (1 + |x|^2)^{\gamma_2} \) with \( 0 < k\gamma_1 + \gamma_2 \leq \alpha \), \( \gamma_1 \geq \frac{\beta - 2}{4} \) and \( k > N + 2 \), then there exists a constant \( C > 0 \) such that

\[
0 < p(x, y, t) \leq C \frac{t^{\sigma}}{(1 + |y|^2)^{-\gamma_2}},
\]

for all \( x, y \in \mathbb{R}^N, 0 < t \leq 1 \) where

\[
\sigma = \frac{2}{\beta - 2} ((k - 2)\gamma_1 + \gamma_2).
\]

**Proof.** Observe that \( W_r(x) = (1 + |x|^2)^r \) is a Lyapunov function for every \( 0 < r \leq \alpha \). If \( \zeta_r(x, t) \) is the corresponding function defined in (2.2.1), then Proposition 2.2.5 yields

\[
\zeta_r(x, t) \leq c_r t^{\frac{2r}{\beta - 2}}
\]

for \( x \in \mathbb{R}^N \) and \( 0 < t \leq 1 \). We set \( a = t \) and \( a - a_0 = b_0 - b = b - a = (1/2)t^s \) where \( s \geq 1 \) will be chosen later and we apply Theorem 3.2.1 with \( \omega(x) = W_1(x) = (1 + |x|^2)^{\gamma_2} \) and \( W_2(x) = (1 + |x|^2)^{k\gamma_1+\gamma_2} \). Thus we obtain

\[
p(x, y, t) \leq C \left( t^{-\frac{2(k\gamma_1+\gamma_2)}{\beta - 2} + s} + t^{-\frac{2\gamma_2}{\beta - 2}} + \frac{s^2}{2} + s \right) (1 + |y|^2)^{-\gamma_2}.
\]

Minimising over \( s \) we get \( s = (4\gamma_1)/(\beta - 2) \) and the thesis follows. \( \square \)

**Example 3.2.5** Let us consider again the operators

\[
A = \Delta - |x|^r \frac{x}{|x|} \cdot D - |x|^s
\]

with \( r > 1 \). Again we distinguishes between three cases.
(a) If \( s + 1 \leq r \), then \( \beta = r + 1 \) and \( \gamma_1 = \frac{r}{2} \). It is easily seen that (3.2.6) holds for every \( \alpha > 0 \) and hence
\[
p(x, y, t) \leq C t^{-(k-2)\frac{r}{r-1} - \frac{2\gamma_1}{r}} (1 + |y|^2)^{-\gamma_2}
\]
for every \( \gamma_2 \geq 0 \), \( 0 < t \leq 1 \), \( y \in \mathbb{R}^N \).

(b) If \( r < s + 1 \), then (3.2.6) holds for \( \beta = s + 2 \) and every \( \alpha > 0 \). So, we have to distinguishes two cases.

(i) If \( s \leq 2r \), then \( \gamma_1 = \frac{r}{2} \) and
\[
p(x, y, t) \leq C t^{-(k-2)\frac{(s+2)\gamma_1}{s} - \frac{2\gamma_1}{s}} (1 + |y|^2)^{-\gamma_2},
\]
(ii) If \( s > 2r \), then \( \gamma_1 = s/4 \) and
\[
p(x, y, t) \leq C t^{-(k-2)\frac{s}{2} - \frac{2\gamma_1}{s}} (1 + |y|^2)^{-\gamma_2},
\]
for every \( \gamma_2 \geq 0 \), \( 0 < t \leq 1 \), \( y \in \mathbb{R}^N \).

**Remark 3.2.6** The results of this section generalize Theorem 4.1 and its corollaries in [23] and also the results obtained in [22] in the case of exponential decay but not for polynomial decay, where the results in [22] are more precise than the one obtained here.

### 3.3 Regularity properties

In this section we obtain the differentiability of the transition semigroup \( T(\cdot) \) associated to the transition kernels \( p \) in \( C_b(\mathbb{R}^N) \) in the case where the coefficients \( F \) and \( V \) are of exponential type.

We assume here that \( a_{ij} \in C^2_b(\mathbb{R}^N) \), \( V \in C^1(\mathbb{R}^N) \) and \( F \in C^2(\mathbb{R}^N) \). All results of this section can be proved exactly by the same arguments as in [23, Section 5 and Section 6].

**Theorem 3.3.1** Suppose that there exist constants \( \beta > 2 \), \( c > 0 \) such that
\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta \beta |x|^{\beta-1}} \right) < -c,
\]
Assume moreover that \( V(x) + |DV(x)| + |F(x)| + |DF(x)| + |D^2F(x)| \leq c_1 \exp(c_2|x|^\beta) \) for some \( \epsilon, c_1, c_2 > 0 \). Then the following estimates hold
(i) \(0 < p(x, y, t) \leq c_3 \exp\{c_4 t^{-\frac{\beta}{2}}\} \exp\{-\gamma|y|^\beta\}\)

(ii) \(|D yp(x, y, t)| \leq c_3 \exp\{c_4 t^{-\frac{\beta}{2}}\} \exp\{-\gamma|y|^\beta\}\)

(iii) \(|D^2 yp(x, y, t)| \leq c_3 \exp\{c_4 t^{-\frac{\beta}{2}}\} \exp\{-\gamma|y|^\beta\}\)

(iv) \(|\partial tp(x, y, t)| \leq c_3 \exp\{c_4 t^{-\frac{\beta}{2}}\} \exp\{-\gamma|y|^\beta\}\)

for suitable \(c_3, c_4, \gamma > 0\) and for all \(0 < t \leq T\) and \(x, y \in \mathbb{R}^N\).

**Remark 3.3.2**

(a) Assuming only that there exist constants \(\beta > 2, c > 0\) such that

\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta^\beta |x|^{\beta-1}} \right) < -c,
\]

and \(V(x) + |F(x)| \leq C \exp(|x|^\gamma)\) for some \(C > 0\) and \(\gamma < \beta\). Then the functions \(p \log^2 p\) and \(p \log p\) are integrable in \(Q(a, b)\) and in \(\mathbb{R}^N\) for fixed \(t \in [a, b]\) respectively and

\[
\int_{Q(a,b)} \frac{|D yp(x, y, t)|^2}{p(x, y, t)} dy dt \leq \frac{1}{\lambda^2} \int_{Q(a,b)} (|F(y)|^2 + V^2(y)) p(x, y, t) dy dt \\
+ \int_{Q(a,b)} p(x, y, t) \log^2 p(x, y, t) dy dt \\
+ \frac{2}{\lambda} \int_{\mathbb{R}^N} [p(x, y, t) - p(x, y, t) \log p(x, y, t)]_{t=a}^{t=b} dy < \infty.
\]

In particular, \(p^\frac{1}{2}\) belongs to \(W^{1,0}_2(Q(a, b))\) (see [23, Theorem 5.1]). This implies in particular that \(p \in W^{2,1}_k(Q(a, b))\) provided that also \(DF\) is of exponential type for some \(k > N + 2\) (see [23, Theorem 5.2]).

(b) From Theorem 3.2.1 and (a) (cf. [23, Theorem 5.3]) one can observe that the assumption \(a_{ij} \in C_k^2(\mathbb{R}^N)\) is not needed for (i) and (ii).

As a consequence we obtain the differentiability of \(T(\cdot)\) in \(C_b(\mathbb{R}^N)\).

**Theorem 3.3.3** Under the assumptions of Theorem 3.3.1, the transition semigroup \(T(\cdot)\) is differentiable on \(C_b(\mathbb{R}^N)\) for \(t > 0\).
Chapter 4

Time dependent Lyapunov functions and kernel estimates

4.1 Integrability of transition kernels

In this section we recall the notion of Lyapunov functions for the elliptic operator $A$ and the parabolic operator $L$ and give conditions under which certain exponentials are Lyapunov functions for $L$. This generalize some results in [29].

Let us recall that the minimal semigroup selects one among all bounded solutions of equation (??), actually the minimal among all positive solutions, when $f$ is positive. The uniqueness of the bounded solution does not hold, in general but it is ensured by the existence of a Lyapunov function, that is of a $C^{2+\alpha}_{\text{loc}}$-function $W : \mathbb{R}^N \to [0, \infty)$ such that $\lim_{|x| \to \infty} W(x) = +\infty$ and $AW \leq \lambda W$ for some $\lambda > 0$ (see [23, Proposition 2.2]). Lyapunov functions are easily found imposing suitable conditions on the coefficients of $A$. For instance, $W(x) = |x|^2$ is a Lyapunov function for $A$ provided that $\sum_i a_{ii}(x) + F(x) \cdot x - |x|^2 V(x) \leq C|x|^2$ for some $C > 0$.

Now, let us introduce, as in [29, Definition 2.2], time depending Lyapunov functions for the parabolic operator $L = \partial_t + A$.

**Definition 4.1.1** We say that a continuous function $W : [0, T] \times \mathbb{R}^N \to [0, \infty)$ is a Lyapunov function for the operator $L$ if it belongs to $C^{2,1}$, $\lim_{|x| \to \infty} W(x, t) = +\infty$ uniformly with respect to $t$ in compact sets of $(0, T]$ and there exist $h : [0, T] \to [0, \infty)$ integrable near 0 such that $LW(x, t) \leq h(t) W(x, t)$ for all $(x, t) \in Q_T$. 

42
Note that the condition \( \lim_{|x| \to \infty} W(x, 0) = \infty \) is not necessary required.

The integrability of time depending Lyapunov functions with respect to the measures \( p(x, y, t) dy \) is given by the following result, which is proved in [29, Proposition 2.3].

**Proposition 4.1.2** For each \( t \in [0, T] \), a Lyapunov function \( W(\cdot, t) \) is integrable with respect to the measure \( p(x, \cdot, t) \). Moreover, if we denote by

\[
\xi_W(x, t) = \int_{\mathbb{R}^N} p(x, y, t) W(y, t) \, dy,
\]

then the inequality

\[
\xi_W(x, t) \leq \exp \left( \int_0^t h(s) \, ds \right) W(x, 0)
\]

holds.

The following result gives a condition under which certain exponentials are time depending Lyapunov functions.

**Proposition 4.1.3** Let \( L = \partial_t + A_0 + F \cdot D - V \) such that there is a constant \( 0 < c < \infty \)

\[
\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \frac{x}{|x|} - \frac{V(x)}{\delta \beta |x|^\beta - 1} \right) < -c
\]

for all \( t \in (0, 1] \) and some constants \( \delta > 0, \beta > 1 \) such that \( \delta < (\beta \Lambda)^{-1}c \). \( \Lambda \) the maximum eigenvalue of \( (a_{ij}) \) as in (H). Then, if \( \alpha > \frac{\beta}{\beta - 2}, \delta < \frac{c}{\Lambda \beta} \) and \( 0 < t \leq 1 \), \( W(x, t) = \exp \{ \delta t^\alpha |x|^\beta \} \) is a Lyapunov function for \( L \). Moreover \( \xi_W(x, t) \leq CW(x, 0) = C \) for some positive constant \( C \) and for all \( x \in \mathbb{R}^N \) and \( 0 < t \leq 1 \).

**Proof.** Let \( W(x, t) = \exp \{ \delta t^\alpha |x|^\beta \}, x \in \mathbb{R}^N, t \in (0, 1] \) and set \( G_i = F_i + \sum_j D_j a_{ij} \). A straightforward computation gives

\[
LW(x, t) = (\partial_t + A_0 + F \cdot D - V) \exp \{ \delta t^\alpha |x|^\beta \}
\]

\[
= \delta \beta t^\alpha W(x, t) \left[ |x|^{\beta - 2} \sum_i a_{ii}(x) + (\beta - 2)|x|^{\beta - 4} \sum_{i,j} a_{ij}(x)x_ix_j + \frac{\alpha}{t^\beta} |x|^{\beta} + \delta \beta t^\alpha |x|^{2(\beta - 2)} \sum_{i,j} a_{ij}(x)x_ix_j + |x|^{\beta - 1} G \cdot \frac{x}{|x|} - \frac{V(x)}{\delta \beta t^\alpha} \right]
\]

43
\[
\begin{align*}
\leq \delta \beta t^\alpha W(x,t) & \left[ |x|^\beta - 2 \Lambda(N + \beta - 2) + \frac{\alpha}{t^\beta} |x|^\beta + \right. \\
& \left. \delta \beta t^\alpha |x|^{2(\beta - 1)} + |x|^{\beta - 1} G \cdot \frac{x}{|x|} - V(x) \right].
\end{align*}
\]

Now, we proceed as in [29].

Let \( \gamma > \frac{1}{\beta - 2} \). If \( |x| > \frac{1}{t^\gamma} \),

\[
LW(x,t) \leq \delta \beta t^\alpha W(x,t) \left[ \frac{\alpha}{\beta} |x|^\beta + \frac{1}{2} + \Lambda(N + \beta - 2) |x|^\beta + \right. \\
\left. \Lambda \delta \beta |x|^{2(\beta - 1)} + |x|^{\beta - 1} G \cdot \frac{x}{|x|} - \frac{V(x)}{\delta t^\alpha} \right],
\]

since \( t \in (0,1] \). By (4.1.3), if \( |x| \) is large enough,

\[
LW(x,t) \leq \delta \beta t^\alpha |x|^{2(\beta - 1)} W(x,t) \left[ \frac{\alpha}{\beta} |x|^{\beta - 1} - \frac{1}{2} + \Lambda(N + \beta - 2) |x|^{\beta - 1} + \Lambda \delta \beta - c \right].
\]

Since \( \delta < (\beta \Lambda)^{-1} c \) and \( \gamma > \frac{1}{\beta - 2} \), for \( |x| \) large enough and belonging to the considered region \( LW \leq 0 \). For the remaining small values of \( x \) in this region \( LW(x,t) \leq C \leq CW(x,t) \). So, in both cases

\[
LW(x,t) \leq CW(x,t) \quad \text{for all } |x| > \frac{1}{t^\gamma} \text{ and } t \in (0,1].
\]

If \( |x| \leq \frac{1}{t^\gamma} \) and is large enough in order that the term containing the drift and the potential is negative, by (4.1.3), then

\[
LW(x,t) \leq W(x,t) \left[ \frac{\alpha \gamma}{\gamma \beta + 1 - \alpha} + \Lambda \delta \beta (N + \beta - 2) \frac{1}{\gamma (\beta - 2) - \alpha} + \Lambda \delta^2 \beta^2 \frac{1}{2^2 \gamma (\beta - 1) - 2 \alpha} \right].
\]

If we choose \( \gamma < \frac{\alpha}{\beta} \), which is possible since \( \alpha > \frac{\beta}{\beta - 2} \), we have \( \gamma \beta - \alpha + 1 < 1 \) and \( 2 \gamma (\beta - 1) - 2 \alpha < 0 \). If \( |x| \) is small we obtain the estimate as in the other region. In both cases we have

\[
LW(x,t) \leq h(t) W(x,t), \quad x \in \mathbb{R}^N, \ t \in (0,1]
\]
with \( h \in L^1(0,1) \). Hence, by (4.1.2),

\[
\xi_w(x, t) \leq \exp \left( \int_0^t h(s) \, ds \right) W(x, 0) \leq C, \quad x \in \mathbb{R}^N, \; t \in [0, 1]
\]

for some constant \( C > 0 \), since \( h \in L^1(0,1) \) and \( W(x, 0) = 1 \).

**Example 4.1.4** If \( L = \partial_t + \Delta - |x|^{r} \cdot D - V \) with \( r > 1 \) and any \( 0 \leq V \in C^\nu_{\text{loc}}(\mathbb{R}^N) \), then Proposition 4.1.3 applies. So, we have for \( \alpha > \frac{\beta}{\beta - 2}, \delta < \frac{1}{\beta - 2} \) and \( 0 < t \leq 1 \), \( W(x, t) = \exp \{ \delta t^\alpha |x|^\beta \} \) is a Lyapunov function for \( L \) and \( \xi_w(x, t) \leq CW(x, 0) = C \) for some positive constant \( C \) and for all \( x \in \mathbb{R}^N \) and \( t \in [0, 1] \).

### 4.2 Pointwise bounds on transition kernels

In this section we apply similar techniques to obtain pointwise bounds. Here we fix \( 0 < a_0 < a < b < b_0 \leq T \) and suppose that \( b_0 - b \geq a - a_0 \).

We consider the following assumption depending on the weight function \( \omega \) which, in our examples, will be an exponential.

\( \text{(H2)} \) Let \( L = \partial_t + A, \; W_1, \; W_2 \) are Lyapunov functions for \( L \), \( W_1 \leq W_2 \) and there exists \( 1 \leq \omega \in C^2 \left( \mathbb{R}^N \times (0, \infty) \right) \) such that for some positive constants \( c_1(a_0, b_0), c_2(a_0, b_0), c_3(a_0, b_0), c_4(a_0, b_0), c_5(a_0, b_0), c_6(a_0, b_0) \) and \( k > N + 2 \)

(i) \( \omega \leq c_1 W_1, \; |D\omega| \leq c_2 \omega^{\frac{k-1}{2}} W_1^{\frac{1}{2}}, \; |D^2\omega| \leq c_3 \omega^{\frac{k-2}{2}} W_1^{\frac{2}{2}}, \; |\partial_t \omega| \leq c_4 \omega^{\frac{k-2}{2}} W_1^{\frac{2}{2}}; \)

(ii) \( \omega |F|^k \leq c_5 W_2 \) and \( \omega V^\frac{k}{2} \leq c_6 W_2 \).

We denote by

\[
\zeta_i(x, t) = \int_{\mathbb{R}^N} p(x,y,t) W_i(y,t) \, dy, \quad \text{for } i = 1, 2.
\]

We use different Lyapunov functions to obtain more precise estimates in the theorem below and its corollary.
Theorem 4.2.1 Assume (H2). Then, there exists a positive constant $C$ such that

$$\omega(y,t)p(x,y,t) \leq C \left[ \left(c_6^k + c_5 + c_3^k + c_2^k c_5 + c_6\right) \int_{a_0}^{b_0} \zeta_2 \, dt + \left(\frac{c_1}{(a-a_0)^{\frac{k}{2}}} + c_4^k\right) \int_{a_0}^{b_0} \zeta_1 \, dt \right]$$

for all $x, y \in \mathbb{R}^N$ and $a \leq t \leq b$.

Proof. Let us assume in the first part of the proof that $\omega$ is bounded. Then, from the previous assumptions and Proposition 4.1.2,

$$\Gamma_1(k, x, a_0, b_0)^k = \int_{Q(a_0, b_0)} \left(1 + |F(y)|^k\right) p(x,y,t) \, dy \, dt \leq \int_{Q(a_0, b_0)} \omega \left(1 + |F(y)|^k\right) p(x,y,t) \, dy \, dt \leq (c_1 + c_5) \int_{Q(a_0, b_0)} p(x,y,t) W_2(y,t) \, dy \, dt \leq (c_1 + c_5) \int_{a_0}^{b_0} \zeta_2(x,t) \, dt < \infty$$

and

$$\Gamma_2(k, x, a_0, b_0)^{k/2} = \int_{Q(a_0, b_0)} V^{\frac{k}{2}}(y) p(x,y,t) \, dy \, dt \leq \int_{Q(a_0, b_0)} \omega V^{\frac{k}{2}}(y) p(x,y,t) \, dy \, dt \leq c_6 \int_{Q(a_0, b_0)} p(x,y,t) W_2(y,t) \, dy \, dt \leq c_6 \int_{a_0}^{b_0} \zeta_2(x,t) \, dt < \infty.$$

From [23, Theorem 4.1], $p \in L^\infty(Q(a,b))$. Let $\eta$ be a smooth function such that $\eta(t) = 1$ for $a \leq t \leq b$, $\eta(t) = 0$ for $t \leq a_0$, $t \geq b_0$, $|\eta'| \leq \frac{2}{a-a_0}$. We consider $\psi \in C^{2,1}(Q_T)$ such that $\psi(\cdot,t)$ has compact support for all $t$. Setting $q = \eta^2 p$ and taking $\varphi(y,t) = \eta^2 \omega(y,t) \psi(y,t)$. Applying (2.2.13) we obtain

$$\int_{Q_T} (\partial_t \varphi(y,t) + A \varphi(y,t)) p(x,y,t) \, dy \, dt = 0$$
and then, after some computations,
\[
\int_{Q_T} \omega q (-\partial_t \psi - A_0 \psi) \, dy dt = \int_{Q_T} q \left( \psi A_0 \omega + 2 \sum_{i,j=1}^{N} a_{ij} D_i \omega D_j \psi + \omega F \cdot D \psi \right) \, dy dt + \\
\int_{Q_T} q (\psi F \cdot D \omega + \psi \partial_t \omega - V \omega \psi) \, dy dt + \\
\int_{Q_T} \frac{k}{2} p \omega \eta^{\frac{k-2}{2}} \partial_t \eta \, dy dt.
\]
Since \( \omega \) is bounded, then \( \omega q \in L^1(Q_T) \cap L^\infty(Q_T) \). Theorem 7.3 in [23] yields
\[
\| \omega q \|_{L^\infty(Q_T)} \leq C \left( \| q D \omega \|_{L^k(Q_T)} + \| q F \omega \|_{L^k(Q_T)} + \| q D^2 \omega \|_{L^\frac{k}{2}(Q_T)} + \| q F D \omega \|_{L^\frac{k}{2}(Q_T)} + \right) + \\
\| q \partial_t \omega \|_{L^\frac{k}{2}(Q_T)} + \| q V \omega \|_{L^\frac{k}{2}(Q_T)} + \frac{1}{a - a_0} \| p \omega \eta^{\frac{k-2}{2}} \|_{L^\frac{k}{2}(Q_T)} \), \quad (4.2.1)
\]
where \( C \) depends on \( N, k, T \) and the \( C^1\)-norm of \( a_{ij} \). Next observe that
\[
\| q F \|_{L^k(Q_T)} = \left( \int_{Q_T} |q F|^k \, dy dt \right)^{\frac{1}{k}} \\
= \left( \int_{Q_T} (\omega q)^{k-1} \omega |F|^k \, dy dt \right)^{\frac{1}{k}} \\
\leq c_2^{\frac{k}{2}} \left( \int_{Q_T} (\omega q)^{k-1} q W_2 \, dy dt \right)^{\frac{1}{k}} \\
\leq c_2^{\frac{k}{2}} \| \omega q \|_{L^\infty(Q_T)}^{\frac{k-1}{k}} \left( \int_{a_0}^{b_0} \zeta_2 \, dt \right)^{\frac{1}{k}}.
\]
In a similar way
\[
\| p \omega \eta^{\frac{k-2}{2}} \|_{L^\frac{k}{2}(Q_T)} \leq c_1^{\frac{2}{k}} \| \omega q \|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left( \int_{a_0}^{b_0} \zeta_1 \, dt \right)^{\frac{2}{k}}; \\
\| q D \omega \|_{L^k(Q_T)} \leq c_2 \| \omega q \|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 \, dt \right)^{\frac{1}{k}}; \\
\| q D^2 \omega \|_{L^\frac{k}{2}(Q_T)} \leq c_3 \| \omega q \|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left( \int_{a_0}^{b_0} \zeta_1 \, dt \right)^{\frac{2}{k}}; \\
\| q \partial_t \omega \|_{L^\frac{k}{2}(Q_T)} \leq c_4 \| \omega q \|_{L^\infty(Q_T)}^{\frac{k-2}{k}} \left( \int_{a_0}^{b_0} \zeta_1 \, dt \right)^{\frac{2}{k}};
\]
47
\[ \|qV\omega\|_{L^\frac{k}{2}(Q_T)} \leq c_6^2 \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{k}{2} \]

and
\[ \|qF \cdot D\omega\|_{L^\frac{k}{2}(Q_T)} \leq c_2 c_5^\frac{k}{2} \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{k}{2}. \]

Therefore, by (4.2.1) and the bounds above,
\[ \|\omega q\|_{L^\infty(Q_T)} \leq C \left[ \left( c_2 + c_5^\frac{k}{2} \right) \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{1}{k} \right] + \left( c_3 + c_2 c_5^\frac{1}{k} + c_6^\frac{2}{k} \right) \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{2}{k} + \left( \frac{c_1^2}{a-a_0} + c_4 \right) \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^\frac{1}{k} \]

and then
\[ \|\omega q\|_{L^\infty(Q_T)}^{\frac{2}{k}} \leq C \left[ \left( c_2 + c_5^\frac{k}{2} \right) \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{1}{k} \right] + \left( c_3 + c_2 c_5^\frac{1}{k} + c_6^\frac{2}{k} \right) \|\omega q\|_{L^\infty(Q_T)} \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{2}{k} + \left( \frac{c_1^2}{a-a_0} + c_4 \right) \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^\frac{1}{k} \].

Setting
\[ A := \left( c_2 + c_5^\frac{1}{k} \right) \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{1}{k} \]
\[ B := \left( c_3 + c_2 c_5^\frac{1}{k} + c_6^\frac{2}{k} \right) \left( \int_{a_0}^{b_0} \zeta_2 dt \right)^\frac{2}{k} + \left( \frac{c_1^2}{a-a_0} + c_4 \right) \left( \int_{a_0}^{b_0} \zeta_1 dt \right)^\frac{1}{k} \]
and
\[ X = \|\omega q\|_{L^\infty(Q_T)}^{\frac{2}{k}}, \]

the inequality (4.2.2) can be written as \( X^2 \leq AX + B \). Hence,
\[ X \leq A + \sqrt{A^2 + 4B} \]

Therefore,
\[ 0 < \omega(y, t) p(x, y, t) \leq C \left[ \left( c_2^k + c_5 + c_3^\frac{k}{2} + c_2^k c_5^\frac{1}{2} + c_6 \right) \int_{a_0}^{b_0} \zeta_2 + \left( \frac{c_1}{(a-a_0)^\frac{k}{2}} + c_4^\frac{k}{2} \right) \int_{a_0}^{b_0} \zeta_1 \right]. \]
If $\omega$ is not bounded, we set $\omega_{\varepsilon} = \omega \frac{\varepsilon}{1 + \varepsilon}$. It is easy to see that $\omega_{\varepsilon}$ is bounded and satisfies $(H2)$ with constants $c_1, c_2, c_3, c_4, c_5, c_6$ independent of $\varepsilon$. Then the estimate of $\|\omega_{\varepsilon}q\|_{L^\infty(Q_T)}$ holds with constants in the right hand side of the previous inequality which do not depend on $\varepsilon$. The claim is now obtained by letting $\varepsilon \to 0$. □

**Corollary 4.2.2** Assume that

$$\limsup_{|x| \to \infty} |x|^{1-\beta} \left( F(x) \cdot \frac{x}{|x|} - \frac{V(x)}{\delta |x|^{|\beta|-1}} \right) < -c, \quad 0 < c < \infty \quad (4.2.3)$$

for some $c > 0, \beta > 2$ and $0 < \delta < (\beta \lambda)^{-1} c$, where $\Lambda$ is the maximum eigenvalue of $(a_{ij})$. If $\alpha > \frac{\beta}{\beta - 2}, K > N + 2$ and $|F| \leq |x|^r, V \leq |x|^s$ for some $r, s > 0$, then

$$0 < p(x, y, t) \leq C \left( \frac{1}{t^{-\frac{\alpha}{\beta}} - 1} + \frac{1}{t^{-\frac{\alpha}{2\beta}} - 1} \right) \exp \left( -\delta \alpha t \beta |y|^\beta \right)$$

for $x, y \in \mathbb{R}^N, 0 < t \leq 1$, for suitable constant $C$.

**Proof.** To verify assumptions $(H2)$ take $\omega = \exp \left( t^\alpha \delta |x|^\beta \right)$, $W_1(x, t) = W_2(x, t) = \exp \left( t^\alpha \gamma |x|^\beta \right)$ with $\delta < \gamma < \frac{\alpha}{\beta}$. By Proposition 4.1.2 we know that $W_1$ is a Lyapunov function for $L$. It is obvious that $\omega \geq 1, \omega \leq W_1$ and so $c_1 = 1$. We have to find $c_2(a_0, b_0)$ such that

$$|D\omega| \leq c_2 \omega^\frac{\alpha - 1}{\beta} W_1^\frac{\alpha}{\beta}. \quad (4.4.4)$$

That is

$$\beta \delta t^\alpha |x|^{\beta - 1} \leq c_2 \exp \left( (\frac{\gamma - \delta}{k}) t^\alpha |x|^\beta \right).$$

Note that

$$\beta \delta t^\alpha |x|^{\beta - 1} = \frac{1}{|x|} \delta \beta \frac{k}{\gamma - \delta} \frac{\gamma - \delta}{k} t^\alpha |x|^\beta \leq \delta \beta \frac{k}{\gamma - \delta} \exp \left( \frac{\gamma - \delta}{k} t^\alpha |x|^\beta \right)$$

for $|x| \geq 1$ and

$$\beta \delta t^\alpha |x|^{\beta - 1} \leq \beta \delta$$
for $|x| < 1$. Then (4.2.4) holds with $c_2 = \delta \beta \max\left\{1, \frac{k}{\gamma - \delta}\right\}$, independent of $a_0$ and $b_0$. By a similar computation we get

$$|D^2\omega| \leq C \left( \beta \delta t^\alpha (\beta - 2 + N) |x|^{\beta - 2} + \beta^2 \delta^2 t^{2\alpha} |x|^{2(\beta - 1)} \right) \omega \leq c_3 \exp\left\{ \frac{2(\gamma - \delta)}{k} t^\alpha |x|^\beta \right\} \omega$$

with $c_3$ not depending on $a_0$ and $b_0$.

For $c_4(a_0, b_0)$, we have

$$|\partial_t \omega| = \alpha t^{\alpha - 1} |x|^\beta \exp\left( t^\alpha \delta |x|^\beta \right)$$

$$= \frac{\alpha k \delta}{2t(\gamma - \delta)} \left( \frac{2(\gamma - \delta)}{k} t^\alpha |x|^\beta \right) \omega$$

$$\leq c_4 \exp\left( \frac{2(\gamma - \delta)}{k} t^\alpha |x|^\beta \right) \omega$$

with $c_4 = \frac{\alpha k \delta}{2(\gamma - \delta) a_0}$. Hence,

$$|\partial_t \omega| \leq c_4 \omega^{\frac{k\alpha - 2}{2(k\alpha - 2)}} W_1^{\frac{k\alpha - 2}{2}}.$$

We have now to find $c_5(a_0, b_0)$ such that

$$\omega |F|^k \leq c_5 W_2.$$

Since $|F| \leq |x|^r$, it suffices to find $c_5$ with

$$\exp\left( t^\alpha \delta |x|^\beta \right) |x|^k r \leq c_5 \exp\left( t^\alpha \gamma |x|^\beta \right).$$

To this purpose let us observe that the function

$$f(s) = s^{kr} \exp\left\{ (\delta - \gamma) t^\alpha s^\beta \right\}$$

attains its maximum for $s = \frac{c(k, \beta, \delta, r)}{t^\beta}$. Therefore $f(s) \leq \frac{c}{t^\beta}$ and we can take

$$c_5 = \frac{c(k, \beta, \delta, \gamma, r)}{a_0}.$$

Finally, we have to find $c_6(a_0, b_0)$ such that

$$\omega V_{\frac{k}{2}} \leq c_6 W_2.$$
From the assumption on $V$ it suffices to find $C_6$ such that
\[
\exp \left( t^\alpha |x|^\beta \right) |x|^{\frac{k\beta}{2}} \leq c_6 \exp \left( t^\alpha |x|^\beta \right).
\]
Similarly, the function
\[
f(z) = z^{\frac{k\beta}{2}} \exp \left\{ (\delta - \gamma) t^\alpha z^\beta \right\}
\]
attains its maximum for $z = \frac{c(k,s,\delta,\gamma)}{t^\frac{\alpha k\beta}{2}}$. Therefore we can take
\[
c_6 = \frac{c(k,s,\delta,\gamma)}{a_0^\frac{2\alpha k\beta}{2}}.
\]
From Theorem (4.2.1), choosing $a_0 = \frac{1}{2} t, a = t, b_0 = \frac{3}{2} t, b = 2 t$ and using Proposition 2.3, we deduce
\[
0 < p(x, y, t) \leq C \left( \frac{1}{t^{\frac{\alpha k\beta}{2} - 1}} + \frac{1}{t^{\frac{\alpha k\beta}{2} - 1}} + \frac{1}{t^{\frac{\alpha k\beta}{2} - 1}} + \frac{1}{t^{\frac{\alpha k\beta}{2} - 1}} \right) \exp \left( -\delta t^\alpha |y| \right)
\]
for all $x, y \in \mathbb{R}^N$ and $0 < t \leq 1$.

Let us now apply the above result.

**Example 4.2.3** We consider the operator
\[
A = \Delta - |x|^r \frac{x}{|x|} \cdot D - |x|^s
\]
with $r > 1$ and assume that $k > N + 2$. For $t \in (0, 1]$ and $x, y \in \mathbb{R}^N$ we have the following three cases.

(i) If $s < 2r$, then $\beta = r + 1$, $\alpha > \frac{r+1}{r-1}$ and $\delta < \frac{1}{r+1}$. Therefore
\[
0 < p(x, y, t) \leq C \frac{t^{\frac{\alpha k\beta}{2} - 1}}{t^{\frac{\alpha k\beta}{2} - 1}} \exp \left( -\delta t^\alpha |y|^{r+1} \right).
\]

(ii) If $s = 2r$, then $\beta = r + 1$, $\alpha > \frac{r+1}{r-1}$ and $\delta < \frac{1+\sqrt{5}}{2(r+1)}$.
(iii) If $s > 2r$, then $\beta = 1 + \frac{s}{2}$, $\alpha > \frac{s+2}{s-2}$ and $\delta < \frac{2}{s+2}$. Then we obtain

$$0 < p(x, y, t) \leq \frac{C}{t^{\frac{s+2}{s-2}}} \exp \left( -\delta t^{\alpha} |y|^{1+\frac{s}{2}} \right) =: c(t) \phi(y, t). \quad (4.2.5)$$

In this case one can also obtain estimates with respect to $x$. In fact, let us consider the formal adjoint $A^* = \Delta + |x|^{\frac{r}{2}} \cdot D + (N + r - 1)|x|^{r-1} - |x|^r$. The associated minimal semigroup has the kernel $p^*(x, y, t) = p(y, x, t)$ which satisfies (4.2.5), by the same argument as above. This yields $p(x, y, t) \leq c(t) \phi(x, t)$. Then we get

$$p(x, y, t) \leq c(t) \phi(x, t)^{\nu} \phi(y, t)^{1-\nu} = \frac{C}{t^{\frac{s+2}{s-2}}} \exp \left( -\nu \delta t^\alpha |x|^{1+\frac{s}{2}} \right) \exp \left( -(1-\nu) \delta t^\alpha |x|^{1+\frac{s}{2}} \right)$$

for any $\nu \in [0, 1]$.

$$p(x, y, t) = \int_{\mathbb{R}^N} p(x, z, \frac{t}{2}) p(z, y, \frac{t}{2}) \, dz \leq c(t/2)^2 \phi(x, t/2) \phi(y, t/2)^{1-\nu} \int_{\mathbb{R}^N} \phi(z, t/2)^\nu \, dz.$$

If we assume moreover that $s > 4$, then one can take $\alpha = 1 + \frac{s}{2}$. Hence, in this case we obtain $\int_{\mathbb{R}^N} \phi(z, t)^\nu \, dz = \int_{\mathbb{R}^N} \exp \left( -\nu \delta t^\alpha |x|^{1+\frac{s}{2}} \right) \, dz = Ct^{-N}$ for some constant $C > 0$. Hence,

$$p(x, y, t) \leq C \frac{c(t/2)^2 \phi(x, t/2) \phi(y, t/2)^{1-\nu}}{t^{-N}}.$$

Therefore,

$$p(x, y, t) = \int_{\mathbb{R}^N} p(x, z, \frac{t}{2}) p(z, y, \frac{t}{2}) \, dz \leq C \frac{c(t/2)^2 \phi(x, t/4) \phi(y, t/2)}{t^{-N}} \int_{\mathbb{R}^N} \phi(z, t/4)^{1-\nu} \, dz \leq C_1 c(t/2)^2 t^{-N} \phi(x, t/4) \phi(y, t/2).$$

$$0 < p(x, y, t) \leq \frac{C}{t^{\frac{r}{2}}} \exp \left( -\nu \delta |tx|^{1+\frac{s}{2}} \right) \exp \left( -(1-\nu) \delta |ty|^{1+\frac{s}{2}} \right)$$

for any $\nu \in [0, 1]$. 

52
Appendix

In this appendix we recall some classical results on PDE’s of elliptic and parabolic problems. Namely, we state some well-known (interior) Schauder estimates. To begin, we introduce the definitions and the main properties of strongly continuous and analytic semigroups of bounded linear operators. Throughout this appendix, $X$ will denote a Banach space.

**Definition 4.2.4** A one-parameter family $T(t)$ of linear operators in $X$ is said to be a semigroup if

$$T(t + s) = T(t)T(s), \ t, s \geq 0.$$  \hspace{1cm} (4.2.6)

**Definition 4.2.5** A semigroup $T(t)$ of linear operators in $X$ is strongly continuous if, for any $x \in X$, $T(t)x$ tends to $x$ as $t$ tends to 0 from the right.

Using the semigroup (4.2.6), one can see that, if $T(t)$ is strongly continuous semigroup, then the function $t \mapsto T(t)x$ is continuous in $[0, +\infty)$ for any $x \in X$. Moreover, it is also possible to show that there exist two constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\|_{L(X)} \leq M \exp(\omega t), \ t > 0.$$  \hspace{1cm} (4.2.7)

When $\omega = 0$ and $M_\omega = 1$, we say that $T(t)$ is a strongly continuous semigroup of contractions in $X$.

We give now the definition of the infinitesimal generator of a strongly continuous semigroup.

**Definition 4.2.6** Let $T(t)$ be a strongly continuous semigroup in $X$. The infinitesimal generator of $T(t)$ is the operator $A : D(A) \subset X \rightarrow X$ defined by

$$D(A) = \{x \in X : \exists g \in X, g = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}\}.$$
\[ Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \]

**Theorem 4.2.7** Assume that the coefficients of \( A \) belong to \( C^\alpha_b(\Omega) \) for some \( \alpha \in (0, 1) \). Further assume that \( u \in C^{1+\frac{2}{N}+\alpha}_{\text{loc}}((0, T) \times \Omega) \) is a bounded (with respect to the sup-norm) solution of the equation

\[ \partial_t u(x, t) = Au(x, t), \quad t \in (0, T), \quad x \in \Omega. \]

Then, there exists a positive constants \( C_1 \), depending only on the coefficients of \( A \), such that

\[ |d^0, u|_\alpha + \sum_{i=1}^N |d, D_i u|_\alpha + \sum_{i,j=1}^N |d^2, D_i D_j u|_\alpha + |d^2, D_t u|_\alpha \leq C_1 \sup_{(0,T) \times \Omega} |u|, \quad (4.2.8) \]

where

\[ |d^m, u|_\alpha = \sup |(d(x, t))^m u(x, t)| + \sup (d(x, t) \wedge d(y, s))^{m+\alpha} \frac{|u(x, t) - u(y, s)|}{(|x-y|^2 + |t-s|)^{\frac{m}{2}}}. \]

In particular, for any open set \( \Omega' \subset \subset \Omega \) and any \( s \in (0, T) \), there exists a positive constant \( C_2 \) depending on \( s \), the coefficients of the operator \( A, \Omega, \Omega' \) and \( T \), such that

\[ \|u\|_{C^{1+\frac{2}{N}+\alpha}(s,T) \times \Omega'} \leq C_2 \sup_{(0,T) \times \Omega} |u|. \quad (4.2.9) \]

Moreover, if \( \text{dist}(\Omega, \Omega') > \sqrt{T} \), then

\[ \sup_{(x,t) \in (0,T) \times \Omega'} (t^{\frac{1}{2}} |Du(x, t)| + t |D^2 u(x, t)|) \leq C_3 \sup_{(x,t) \in (0,T) \times \Omega} |u(x, t)|, \quad (4.2.10) \]

for some positive constant \( C_3 \), depending only on the coefficients of \( A, \Omega, \Omega' \) and \( T \).

We present also a simple, purely analytical, proof of the embedding of the spaces \( H^{k,1}(Q_T) \), due to Krylov, see [14]. Krylov's proves the above embedding for stochastic parabolic Sobolev spaces. The same method proves an embedding for the spaces \( V^k(Q_T) \) which we used in Chapter 2.

**Lemma 4.2.8** There exist linear, continuous extension operators \( E_1 : H^{k,1}(Q_T) \to H^{k,1}(\mathbb{R}^{N+1}) \) and \( E_2 : V^k(Q_T) \to V^k(\mathbb{R}^{N+1}) \).

**Proof.** The proof is easily achieved using standard reflection arguments with respect to the variable \( t \). \( \square \)
Lemma 4.2.9  The restrictions of functions in $C_c^\infty(R^{N+1})$ to $Q_T$ are dense in $\mathcal{H}^{k,1}(Q_T)$ and in $\mathcal{V}^k(Q_T)$.

Proof. If $u \in H^{k,1}(Q_T)$ we consider $v = Eu \in \mathcal{H}^{k,1}(R^{N+1})$. By standard arguments involving convolutions and multiplications with cut-off functions, we may approximate $v$ with smooth functions with compact support in the norm of $\mathcal{H}^{k,1}(R^{N+1})$, hence $u$. The proof for $\mathcal{V}^k(Q_T)$ is similar. □

Theorem 4.2.10  (i) If $1 < k < N + 2$, then $H^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $1/r = 1/k - 1/(N+2)$.

(ii) If $k = N + 2$, then $H^{k,1}(Q_T)$ is continuously embedded in $L^r(Q_T)$ for $N + 2 \leq r < \infty$.

(iii) If $k > N + 2$, then $H^{k,1}(Q_T)$ is continuously embedded in $C_0(Q_T)$.

Proof. By Lemma 4.2.8, 4.2.9 it is sufficient to establish the estimate

$$\|u\|_{L^r(Q)} \leq C\|u\|_{\mathcal{H}^{k,1}(Q)},$$

for every $u \in C_c^\infty(Q)$, with $C$ independent of $u$, where $Q = R^{N+1}$ and $r$ is as in (i), (ii) or $r = \infty$ in case (iii).

We consider the fundamental solution $G$ of the operator $I + \partial_t - \Delta$ in $Q$. We have

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{N/2}} \exp \left( -\frac{1}{4t} |x|^2 - t \right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Let $u, \psi \in C_c^\infty(Q)$ and consider $\phi = G * \psi$. The function $\phi$ belongs to $C^2(Q)$ and satisfies $\phi + \partial_t \phi - \Delta \phi = \psi$, see e.g. [12, Theorem 8.4.2]. By a straightforward computation one sees that $G \in L^s(Q)$ for $1 \leq s < (N+2)/N$ and $D_x G \in L^s(Q)$ for $1 \leq s < (N+2)(N+1)$. Young’s inequality then yields $\|\phi\|_{W_s^{1,1}(Q)} \leq c_1\|\psi\|_{L^1(Q)}$ for $s < (N+2)/(N+1)$. Since $k > N + 2$, $k' < (N+2)/(N+1)$ and we get

$$\left| \int_Q u \psi dx dt \right| = \left| \int_Q u(\phi + \partial_t \phi - \Delta \phi) dx dt \right| = \left| \int_Q u(\phi + \partial_t \phi) + Du \cdot D\phi dx dt \right| \leq c_2\|u\|_{\mathcal{H}^{k,1}(Q)}\|\phi\|_{W_{k'}^{1,1}(Q)} \leq c_3\|u\|_{\mathcal{H}^{k,1}(Q)}\|\psi\|_{L^1(Q)}.$$
In order to prove (ii) we fix $N + 2 < r < \infty$ and choose $1 < s < (N + 2)/(N + 1)$ such that
\[ \frac{1}{k'} = \frac{1}{s} + \frac{1}{r'} - 1. \]
Young’s inequality then yields $\|\phi\|_{W^{1,0}_r(Q)} \leq c_1 \|\psi\|_{L^{r'}(Q)}$ hence
\[ \left| \int_Q u\psi \, dx \, dt \right| \leq c \|u\|_{H^{k,1}(Q)} \|\psi\|_{L^{r'}(Q)} \]
and (ii) is proved.

To prove (i) we use the estimate $\|\phi\|_{W^{2,1}_r(Q)} \leq c \|\psi\|_{L^{r'}(Q)}$, see [19, Theorem 9.2.3] and the embedding $W^{2,1}_r(Q) \subset W^{1,0}_{k'}(Q)$, see [19, Lemma II.3.3] to conclude as before.

A closer look at the above proof shows an embedding of the space $V^k(Q_T)$, used in Chapter 3.

**Theorem 4.2.11** If $k > N + 2$, then $V^k(Q_T)$ is continuously embedded in $C_0(Q_T)$.

**Proof.** As above we may assume that $Q_T = Q$. Choose $\varphi, \psi$ as in the above theorem. Then
\[ \left| \int_Q u\psi \, dx \, dt \right| = \left| \int_Q u(\varphi + \partial_t \varphi - \Delta \varphi) \, dx \, dt \right| = \left| \int_Q u(\varphi + \partial_t \varphi) + Du \cdot D\varphi \, dx \, dt \right| \leq \|u\|_{V^k(Q)} \left( \|D\phi\|_{L^\infty(Q)} + \|\phi\|_{L^\infty(Q)} \right) \leq c \|u\|_{V^k(Q)} \|\psi\|_{L^1(Q)} \]
by the above estimates for $\varphi$, since $k/(k-1) < (N+2)/(N+1)$ and $k/(k-2) < (N + 2)/N$. 

We need the following estimate of the sup norm of solution of parabolic problems.

**Theorem 4.2.12** Let $k > N + 2$, $v \in L^k(Q_T)$, $w \in L^2(Q_T)$ and assume that $u \in L^k(Q_T) \cap L^k(Q_T)$ satisfies
\[ \int_{Q_T} u(\partial_t \varphi + A_0 \varphi) \, dx \, dt = \int_{Q_T} (v \cdot D\varphi + w\varphi) \, dx \, dt \]
for every $\phi \in C^{2,1}(Q_T)$ such that $\phi(\cdot, t)$ has a compact support for every $t$. Then

$$
\|u\|_{L^\infty(Q_T)} \leq C \left( \|u\|_{L^\frac{3}{2}(Q_T)} + \|u\|_{L^1(Q_T)} + \|v\|_{L^1(Q_T)} + \|w\|_{L^\frac{3}{2}(Q_T)} \right)
$$

where $C$ depends on $N, T, k$ and the $C^1_k$-norm of $a_{ij}$.

**Proof.** We have

$$
\int_{Q_T} u(\partial_t \phi + A_1 \phi) \, dx \, dt = \int_{Q_T} (g \cdot D\phi + V \phi) \, dx \, dt,
$$

where $A_1 = \sum_{i,j} a_{ij} D_{ij}$ and $g_i = v_i + u D_i(\sum_{j=1}^N a_{ij})$ and therefore

$$
\left| \int_{Q_T} u(\partial_t \phi + A_1 \phi) \, dx \, dt \right| \leq C \left( \|u\|_{L^1(\partial Q_T)} + \|v\|_{L^1(\partial Q_T)} \right)\|D\phi\|_{L^\frac{3}{2}(Q_T)} + \|w\|_{L^\frac{3}{2}(Q_T)}\|\phi\|_{L^\frac{3}{2}(Q_T)}
$$

Replacing $\phi$ by its difference quotients with respect to the variable $x$ we obtain

$$
\left| \int_{Q_T} \tau_h u(\partial_t \varphi + A_1 \phi) \, dx \, dt \right| \leq C \left( \|u\|_{L^1(\partial Q_T)} + \|v\|_{L^1(\partial Q_T)} \right)\|\phi\|_{W^{2,1}_k(Q_T)} + \|w\|_{L^\frac{3}{2}(Q_T)}\|\phi\|_{L^\frac{3}{2}(Q_T)}
$$

By Sobolev embedding

$$
\|D\phi\|_{L^s(Q_T)} \leq C\|\phi\|_{W^{2,1}_k(Q_T)}
$$

if $1/s = 1 - 1/k - 1/(N+2)$ (note that $k/(k-1) < N+2$). Since $k/(k-1) < k/(k-2) < s$, because $k > N+2$, we can estimate the $L^k/(k-2)$ norm of $D\phi$ with its $W^{2,1}_k/(k-1)$ norm thus obtaining

$$
\left| \int_{Q_T} \tau_h u(\partial_t \varphi + A_1 \phi) \, dx \, dt \right| \leq C \left( \|u\|_{L^1(\partial Q_T)} + \|v\|_{L^1(\partial Q_T)} + \|w\|_{L^\frac{3}{2}(Q_T)} \right)\|\phi\|_{W^{2,1}_k(\partial Q_T)}
$$

We approximate $\phi$ in $W^{2,1}_k/(k-1)(Q_T)$ with a sequence of functions $\varphi_n \in C^{1,2}(Q_T)$ with compact support in the space variables. Since $u \in L^k(Q_T)$, writing the above inequality for $\varphi_n$ and letting $n \to \infty$ we see that it holds for $\phi$. 

57
This yields \( u \in W^{1,0}_k(Q_T) \) and
\[
\|D u\|_{L^k(Q_T)} \leq C \left( \|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^k_2(Q_T)} \right)
\]
Now we treat the time derivative. We have
\[
\int_{Q_T} u \partial_t \phi \, dx \, dt = \int_{Q_T} \left( \sum_{i,j} a_{ij} D_i u D_j \phi + v \cdot D\phi + w \phi \right) \, dx \, dt
\]
and hence, using the above estimates,
\[
\left| \int_{Q_T} u \partial_t \phi \, dx \, dt \right| \leq C \left( \|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^k_2(Q_T)} \right) \|D\phi\|_{L^\frac{k}{k-1}(Q_T)}
+ \|w\|_{L^\frac{k}{2}(Q_T)} \|\phi\|_{L^\frac{k}{k-2}(Q_T)}
\]
Since also \( u \in L^\frac{k}{2}(Q_T) \) we deduce that \( u \in V^k(Q_T) \) and hence Theorem 4.2.11 yields \( u \in L^\infty(Q_T) \) and
\[
\|u\|_{L^\infty(Q_T)} \leq C \|u\|_{V^k(Q_T)} \leq C \left( \|u\|_{L^\frac{k}{2}(Q_T)} + \|u\|_{L^k(Q_T)} + \|v\|_{L^k(Q_T)} + \|w\|_{L^k_2(Q_T)} \right).
\]
\( \square \)
Bibliography


